THE VOLUME OF AN n-SIMPLEX WITH MANY EQUAL EDGES

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It is well known that the volume of a regular n-simplex with edge length s is

$$\frac{s^n}{n!}\sqrt{\frac{n+1}{2^n}}\ .$$

But suppose one edge has length b and all the other edges have length a. Is there a simple formula for the volume of the simplex in that case? What if all the edges incident at a given vertex have length b and all the other edges have length a?

It is these questions that motivated the investigation that led to the following result:

Theorem. Let K be an n-simplex in E^n. Suppose the vertices of K are colored with r colors, c_1, c_2, ..., c_r $(1 \le r \le n+1)$. Let the number of vertices colored c_i be m_i $(1 \le m_i \le n+1)$. It is given that if an edge has both its vertices the same color, c_i, the length of that edge is a_i. If the two vertices of an edge have different color, the edge has length s. Then the volume of K is

$$\frac{1}{n!\,2^{\frac{n}{2}}}\prod_{i=1}^{r} a_i^{m_i-1}\sqrt{(-1)^{r+1}\left(\prod_{i=1}^{r}\left((m_i-1)a_i^2-m_is^2\right)\right)\sum_{i=1}^{r}\frac{m_i}{(m_i-1)a_i^2-m_is^2}}\ .$$

Proof. The volume, V, of an n-simplex in terms of the edge lengths, $\{a_{ij}\}$, is determined by the formula

$$(-1)^{n+1}2^n(n!)^2V^2 = D \tag{1}$$

where D is given by the determinant

$$\begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & \cdots & a_{1n}^2 & a_{1,n+1}^2 & 1 \\ a_{21}^2 & 0 & a_{23}^2 & \cdots & a_{2n}^2 & a_{2,n+1}^2 & 1 \\ a_{31}^2 & a_{32}^2 & 0 & \cdots & a_{3n}^2 & a_{3,n+1}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n+1,1}^2 & a_{n+1,2}^2 & a_{n+1,3}^2 & \cdots & a_{n+1,n}^2 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}.$$

(See [1] for a proof.)

Now, let us assign the edge lengths as specified in the theorem, except that to make the computations simpler, let us assume the edge lengths are $\sqrt{a_i}$ and $\sqrt{s}$ (instead of a_i and s). A simple transformation then will change the result we get into the form required by the statement of the theorem.

We find that the resulting determinant consists of r square blocks along the main diagonal and the last row and column being the same as shown above. The ith block has the form

$$\begin{pmatrix} 0 & a_i & a_i & a_i & \cdots & a_i & a_i \\ a_i & 0 & a_i & a_i & \cdots & a_i & a_i \\ a_i & a_i & 0 & a_i & \cdots & a_i & a_i \\ a_i & a_i & a_i & 0 & \cdots & a_i & a_i \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_i & a_i & a_i & a_i & \cdots & 0 & a_i \\ a_i & a_i & a_i & a_i & \cdots & a_i & 0 \end{pmatrix}$$

and every other element in the determinant has value s. For example, if $n = 11$, $r = 3$, $a_1 = a$, $m_1 = 4$, $a_2 = b$, $m_2 = 5$, $a_3 = c$, and $m_3 = 3$, then the determinant is as follows:

$$\begin{vmatrix} 0 & a & a & a & s & s & s & s & s & s & s & s & 1 \\ a & 0 & a & a & s & s & s & s & s & s & s & s & 1 \\ a & a & 0 & a & s & s & s & s & s & s & s & s & 1 \\ a & a & a & 0 & s & s & s & s & s & s & s & s & 1 \\ s & s & s & s & 0 & b & b & b & b & s & s & s & 1 \\ s & s & s & s & b & 0 & b & b & b & s & s & s & 1 \\ s & s & s & s & b & b & 0 & b & b & s & s & s & 1 \\ s & s & s & s & b & b & b & 0 & b & s & s & s & 1 \\ s & s & s & s & b & b & b & b & 0 & s & s & s & 1 \\ s & s & s & s & s & s & s & s & s & 0 & c & c & 1 \\ s & s & s & s & s & s & s & s & s & c & 0 & c & 1 \\ s & s & s & s & s & s & s & s & s & c & c & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} .$$

We now proceed to evaluate this determinant by applying elementary row and column operations. In each group of m_i rows ($i = 1, \ldots, r$), we subtract every row (except the last row) from the row above it. Then, in each group of m_i columns, we subtract each column (except the last column) from the column to its left. We wind up with a matrix where each square block along the diagonal has been replaced by a matrix whose diagonal entries are all $-2a_i$, (except for the lower right entry with value 0), and whose minor diagonals just below and above the main diagonal all have value a_i. Furthermore, all the s entries have disappeared with the exception of those whose rows and columns are at the end of the groups of m_i. The 1's in the last row and column have also turned to 0's except those occurring at the ends of groups of m_i entries.

In our example, the resulting determinant is

$$\begin{vmatrix}
-2a & a & & & & & & & & & & & 0 \\
a & -2a & a & & & & & & & & & & 0 \\
 & a & -2a & a & & & & & & & & & 0 \\
 & & a & 0 & & & & & s & & & s & 1 \\
 & & & & -2b & b & & & & & & & 0 \\
 & & & & b & -2b & b & & & & & & 0 \\
 & & & & & b & -2b & b & & & & & 0 \\
 & & & & & & b & -2b & b & & & & 0 \\
 & & & s & & & & b & 0 & & & s & 1 \\
 & & & & & & & & & -2c & c & & 0 \\
 & & & & & & & & & c & -2c & c & 0 \\
 & & & s & & & & & s & & c & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{vmatrix}.$$

All missing elements in the display are 0's.

In each square block, we can remove the a_i's situated along the two minor diagonals by adding in the appropriate multiple of the preceding row or column, in succession, top to bottom and left to right. In our example, we would multiply the first row by 1/2 and add it to the second row, then multiply the first column by 1/2 and add it to the second column. This leaves us with $(\frac{1}{2} - 2)a = -\frac{3}{2}a$ in row 2 column 2. Thus, we multiply row 2 by 2/3 and add it to row 3. Then we multiply column 2 by 2/3 and add it to column 3. This leaves us with $-\frac{4}{3}a$ in row 3 column 3, etc.

In general, the multipliers will be 1/2, 2/3, 3/4, . . . , $(m_i - 1) / m_i$. The final numbers along the main diagonal will be $-2a_i/1$, $-3a_i/2$, $-4a_i/3$, . . . , $-m_ia/(m_i - 1)$, $(m_i - 1)a/m_i$.

In our example, we get

$$\begin{vmatrix}
\frac{-2a}{1} & & & & & & & & & & & & 0 \\
& \frac{-3a}{2} & & & & & & & & & & & 0 \\
& & \frac{-4a}{3} & & & & & & & & & & 0 \\
& & & \frac{3a}{4} & & & & & s & & & s & 1 \\
& & & & \frac{-2b}{1} & & & & & & & & 0 \\
& & & & & \frac{-3b}{2} & & & & & & & 0 \\
& & & & & & \frac{-4b}{3} & & & & & & 0 \\
& & & & & & & \frac{-5b}{4} & & & & & 0 \\
& & & s & & & & & \frac{4b}{5} & & & s & 1 \\
& & & & & & & & & \frac{-2c}{1} & & & 0 \\
& & & & & & & & & & \frac{-3c}{2} & & 0 \\
& & & s & & & & & s & & & \frac{2c}{3} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{vmatrix}.$$

Most of the entries on the major diagonal now have all 0's in their rows. We can thus expand the determinant by minors along these rows and see that the value of the determinant is

$$\prod_{i=1}^{r}(-a_i)^{m_i-1}m_i$$

times the following determinant

$$\begin{vmatrix}
\frac{(m_1-1)a_1}{m_1} & s & s & \cdots & s & 1 \\
s & \frac{(m_2-1)a_2}{m_2} & s & \cdots & s & 1 \\
s & s & \frac{(m_3-1)a_3}{m_3} & \cdots & s & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s & s & s & \cdots & \frac{(m_r-1)a_4}{m_r} & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{vmatrix}.$$

This determinant is simplified by subtracting s times the bottom row from every other row. We are left with a determinant whose last row and column are all 1's (except for the 0 in the lower right corner). The remaining elements all lie along the main diagonal, and are $\frac{(m_i-1)a_i}{m_i} - s, i = 1, 2, \ldots, r$. In our example, this comes out to

$$\begin{vmatrix} \frac{3a}{4} - s & 0 & 0 & 1 \\ 0 & \frac{4b}{5} - s & 0 & 1 \\ 0 & 0 & \frac{2c}{3} - s & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} .$$

Finally, this determinant is evaluated by getting rid of the 1's in the final row. To do that, multiply each of the first r rows by the reciprocal of the diagonal element and subtract the result from the last row. This changes the 1's in the last row to 0's and changes the 0 to

$$-\sum_{i=1}^{r} \left(\frac{(m_i - 1)a_i}{m_i} - s \right)^{-1} .$$

The determinant is now upper triangular and so its value is the product of the diagonal elements. We have thus found that

$$D = \prod_{i=1}^{r} (-a_i)^{m_i - 1} \big((m_i - 1)a_i - m_i s \big) \left(- \sum_{i=1}^{r} \frac{m_i}{(m_i - 1)a_i - m_i s} \right) .$$

Comparing this with formula (1) and noting that $\sum_{i=1}^{r} m_i = n+1$, we see that we can move the $(-1)^{n+1}$ to the right hand side and wind up with $(-1)^{r+1}$. Then, solving for V^2 and taking the square root of both sides proves our theorem.

Letting $r = 2$ gives us two interesting corollaries.

Corollary 1. An n-simplex in E^n $(n \geq 1)$ has one edge of length b. Every other edge has length a. Then the volume of the simplex is

$$\frac{ba^{n-2}}{n!2^{\frac{n}{2}}}\sqrt{2na^2-(n-1)b^2}\ .$$

Corollary 2. An n-simplex in E^n $(n \geq 1)$ has every edge incident at a given vertex of length a. Every other edge has length b. Then the volume of the simplex is

$$\frac{b^{n-1}}{n!2^{\frac{n}{2}}}\sqrt{2na^2-(n-1)b^2}\ .$$

References

1. D. M. Y. Sommerville, *An Introduction to the Geometry of n Dimensions.* Dover Publications, Inc. New York, 1958.