A PUZZLE IN KELLEY'S APPENDIX

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In the appendix of John L. Kelley's General Topology there is a system of axiomatic set theory consisting of eight axioms and an axiom scheme from the unpublished lecture notes of Anthony P. Morse. Its presence there poses a puzzle: What is the relation of axiomatic set theories to general topology? Is it the same as that of Cantorian set theory? If not, what are the differences and their consequences for general topology? The answers to these questions will be given here after this sytem has been compared to the two standard systems of axiomatic set theory, Zermelo-Fraenkel (abbreviated ZF) and von Neumann-Bernays-Gödel (abbreviated VBG) and the metamathematics of these theories has been discussed in detail.

These set theories are first order formal theories. A formal theory consists of three parts, specific rules for the formation of its language in the setting of symbolic logic, rules for the formation of sentences (formulas) within this language, and rules for the formation of theorems as deduced from its axioms. A first order theory deals with one kind of object only. In these theories that object is either a set or class. These theories are embedded in first order logic which gives them an artificial aspect since one formal theory is then developed within another. This artificialness has great consequences for these theories.

These axiomatic set theories differ from Cantorian set theory in another important respect. Only the relationships among sets or classes are investigated. Thus, they are structural theories in which elements that are not sets or classes are ignored. Class, as used here, means a collection that is not necessarily a set.

The system ZF utilizes one primitive term, set, and eight axioms. If the axiom of choice is included, the new system is known as ZFC. There is one binary, predicate, ε, and equality is defined as: $x = y$ if for all z, $z \varepsilon x$ if and only if $z \varepsilon y$. All sentences are formed from $x \varepsilon y$ or $x = y$ by negation, conjunction, disjunction, and the use of universal and existential quantifiers. A sentence is open in a variable x if it occurs at least once unquantified. A condition is an open sentence in x. The above considerations apply to the other two set theories as well, subject to the stipulation that if the primitive term is class, that equality of classes is as above with class in place of set.

The axioms of ZF are:

1) Extension: If $x = y$, then for all w, if $x \varepsilon w$, then $y \varepsilon w$.
2) Pairing: For any two distinct sets x and y, there is a set consisting of just these two sets.
3) Union: For any set a which contains at least one element, there is a set whose members are exactly those members of elements of a.
4) Power set: For any set a there is the set whose elements are exactly the subsets of a.
5) Infinity: There is at least one set Z such that $\phi \varepsilon z$ and if $x \varepsilon z$, then $\{x\} \varepsilon z$.
6) Schema of Replacement: For any set S and any single valued predicate f with free (unquantified) variable x, there is the set of exactly those elements $f(x)$ with $x \varepsilon S$.
7) Subsets: For any set a and any condition $P(x)$ on x there is the set that consists exactly of those elements of a that satisfy $P(x)$.
8) Regularity: Every nonempty set a contains at least one element b such that a and b have no element in common.

To obtain ZFC add:

9) Choice: For each set a there is a function whose domain is the collection of nonempty sets of a and for each such set b, $f(b) \varepsilon b$.

These axioms are not independent. The schema (6) implies both (2) and (7). From (7) can be deduced the existence of the empty set, unit set, the intersection of two sets, and the cartesian product of the elements of a set. Axiom (5) is needed to guarantee the existence of infinite sets, (6) is needed to develop a general theory of ordinal numbers including transfinite induction, and (8) disallows $x \varepsilon x$ and places the sets in layers creating a theory of types.

In VBG there are two primitive terms, set and class. This is in appearance only since set is defined in terms of class. This system retains (1), (2), (3), (4), (5), and (8) but (1) and (8) are stated for classes. There is a set of eight axioms that allow for the existence of a class A consisting of exactly those elements x which satisfy $P(x)$, where $P(x)$ is a condition that does not contain quantifiers over classes. In addition axiom (6) is replaced by assuming that for any set x and any single-valued A, there is a set y whose elements are just those sets which bear the relation defined by A to the members of x. From this axiom and the class theorem may be obtained each instance of (6). There are two other axioms central to this sytem:

1) every set is a class, and
2) if $x \varepsilon y$, then x is a set where x and y are classes.

Axiom (9) is not part of VBG but Gödel stated it for classes and included it for reference since he wanted to prove that it was consistent with the other axioms of this sytem.

In Morse's system (abbreviated QM) the one primitive term is class. In this system the statment "If $x \varepsilon y$, then x is a set," is, of course, a definition. The system retains (1), (3), (4), (5), (8), and adds (9). Axioms (1), (8), and (9) are stated for classes. Axiom (6) is handled as in VBG. That is, it is replaced by a single axiom which was given above. Axiom (2) is replaced by: If x and y are sets, then $x \cup y$ is a set. This axiom implies (2). Finally, there is the Classifier Axiom-Scheme. After preliminaries stating under what conditions sentences (formulas) can be formed using the classifier, $\{x\varepsilon X\}$, it is stated as follows:

> An axiom results if in the following 'α' and 'β' are replaced by variables, 'A' by a formula and 'B' by the formula obtained from by replacing each occurrence of the variable which replaced α by the variable which replaced β:
>
> For each β, $\beta\varepsilon\{\alpha : A\}$ if and only if β is a set and B.

This is the classifier analogue of the definition of set and allows classes to be predicated by assuming the existence of a class A which consists exactly of those elements x which satisfy $P(x)$, where $P(x)$ is any condition. Since this is stronger than the class theorem in VBG, it may be used to obtain each instance of (6).

A theory of classes may be developed in ZF by extending the language of the system and endowing class abstracts, $\{x|P(x)\}$ with the desired properties. Thus, there are three distinct theories of classes. In QM $\{x|P(x)\}$ always represents a class (as a concept apart from formalism) but this is not so in the other two theories.

It is not known whether any of these systems are consistent. However, ZF and VBG are equiconsistent and the consistency of QM implies the consistency of the other two systems. Moreover, every theorem ZF is a theorem of VBG, and any theorem of VBG concerned only with sets is a theorem of ZF. On the other hand, there are an infinite number of theorems in QM not provable in either ZF or VBG.

None of these systems are categorical. One way to show this is by using the Löwenheim-Skolem Theorems. The first states that in first order logic, if a set of propositions has a model, it has a countable model (in axiomatic set theories, a denumerable model) and the second that under the same hypotheses, that if the set of propositions has a model of some infinite cardinality, it also has a model of every larger cardinality.

In the case of axiomatic set theories the first theorem is known as the Skolem Paradox. It is so called since within each of the theories we may prove the existence of a nondenumerable infinite set. That it is actually not a paradox follows from axiom (7) (and the predication of classes in the other sytems) since sets (or classes) may be predicated in at most a denumerable number of ways.

Gödel's First Incompleteness Theorem provides a second method of demonstrating the noncategoricalness of systems. Let Z be the system of arithmetic that is based on Peano's axioms and the usual recursive definitions of addition and multiplication. Then if S is a system that contains Z and is consistent, there exists a statement of the form "for all $n, G(n)$" which is undecidable in S although $G(0), G(0')$, — are provable in S. Here, $0'$ is the successor of 0. The noncategoricalness then follows from two considerations. First, each of the systems is as above, and second, if the axiom system is categorical then it is complete. Not only is such a system S incomplete but it is incompletable.

This famous theorem, proved by Gödel in 1931, goes to the heart of axiomatic set theories. Gödel demonstrated in 1936 that the axiom of choice, the continuum hypothesis, and the generalized continuum hypothesis are consistent with ZF. In 1963, Paul Cohen, using his forcing technique proved that the negations of these statements are consistent with ZF. Therefore, each of these statements is undecidable in ZF. This means that if any of them are adopted as axioms, they are independent of those of ZF. This precipitated a crisis in the mathematical community since it shows the inadequacy of axiomatic theories. Worse was yet to come! The following statements from analysis, algebra, and topology have shown, using Cohen's technique to be consistent with ZF or VBG:

a) R is the countable union of countable sets. If this is accepted as an axiom, it can be proved that Lebesque measure is not countably additive.

b) There exists a free group whose commutator subgroup is not free. It is known that any subgroup of a free group is free!

c) The negation of Urysohn's Lemma.

Each of these statements is a candidate for an undecidable statement in ZF or VBG.

The incompleteness of ZF may be exploited to demonstrate its relative consistency. The sentence stating the existence of a least inaccessible cardinal α is undecidable in ZFC. Form a new axiom system consisting of ZFC and the negation of this axiom. If $R(\alpha)$ is defined by $R(0) = \phi$ and $R(\alpha) = \bigcup_{\beta<\alpha}(P(R(\beta)))$ where α and β are ordinals, then $R(\alpha)$ is a model for this new system. $P(R(\beta))$ is the power set of $R(\beta)$. Hence, ZFC is consistent relative to this new system. Since the continuum hypothesis is undecidable in ZF, ZF is consistent relative to ZFC and this new system.

Finally, using recursive function theory, it can be shown that none of these systems develop arithmetic in its customary form. Nor is it possible to develop category theory in them. However, it is possible to modify them to handle this discipline.

At last we can examine the relationship of axiomatic set theories to general topology. The applications of ZF, or equivalently by VBG, to general topology depend on the use of cardinal functions. Two examples are the weight and Lindelöff number of a topological space. The weight of a topological space is the least cardinality of a basis for the space. A second countable space has weight ω where ω is finite or denumerable. The Lindelöff number of a topological space is the least cardinal α such that every open cover of the space has a subcover of cardinality not exceeding α. Compact and Lindelöff spaces have Lindelöff number ω. These and related functions are most useful in problems of metrization, normality, and compactification. The tools for applying such ideas are the usual methods of axiomatic set theory: forcing, trees, and models. The useful models are ZF with Martin's axiom and the negation of the continuum hypothesis and Gödel's constructible universe.

Martin's Axiom states that the following is true for $c \leq \kappa < 2^{\omega}$ where ω is the first infinite cardinal: $ax(\kappa)$: If P is a nonempty partially ordered set satisfying the countable chain condition (CCC), and $\{Di|i\varepsilon I\}$ is a family of dense subsets of P having cardinality not exceeding κ, then there is a subnet Q of P which intersects each Di.

D is dense in P if for every $p\varepsilon P$ there is a $q\varepsilon D$ such that $p \leq q$. If p and q are in P they are said to be incompatible if there is no $r\varepsilon P$ with $p \leq r$ and $q \leq r$. P satisfies the CCC if every pairwise incompatible subset of P is countable.

This axiom is known to be true if $\kappa = \omega$ so that Martin's Axiom generalizes this result. The continuum hypothesis and ZF imply Martin's Axiom. Its topological form is that no compact CCC Hausdorff space is the union of less than c nowhere dense sets. Assuming the continuum hypothesis, this form of the axiom is a form of the Baire Category theorem. We can deduce if we have ZF, the negation of the continuum hypothesis, and this axiom that every product of CCC spaces is CCC. From $ax(\kappa)$ with ZF it can be shown that if a topological space has a countable base, then the union of not more than κ meager sets is meager and the product of not more than κ compact metric spaces is sequentially compact.

The following questions, given by Mary Ellen Rudin were open questions in 1977:

1) Is there a normal nonmetrizable image of a metric space under a compact open map?
2) Is every normal Moore space compact?

All spaces are Hausdorff. With ZF, the negation of the continuum hypothesis, and Martin's Axiom the first is known to be true and the second false. This goes to the heart of the matter. Axiomatic set theories are of service to general topology by suggesting avenues by which such questions might be answered without utilizing it. In other words, it helps us formulate useful conjectures.

In the first of his seven papers on axiomatic set theory, Paul Bernays observed that which system of set theory one uses depends on what use you are to make of it. Since the time of Cohen's forcing method it is clear that the uses are to demonstrate consistency and independence results. It is these results that find application in general topology.

Three axiomatic set theories have been examined here. There are many others. As Quine has observed, there are radically inequivalent axiomatic set theories and there is no optimal one. This is due to the great equalizer, Gödel's First Incompleteness Theorem. Such theories demonstrate the limits of the axiomatic method. Thus, axiomatic set theories are very unlike Cantorian set theory both in content and applications. Nowhere is this clearer than in their connection with general topology.

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