Let $p_1 = 2, p_2 = 3, \ldots$ be the sequence of primes. We will need the following two lemmas.

LEMMA 1. $\prod_{i=1}^{m}(1 - 1/p_i) \geqslant 1/p_m, m = 1, 2, \ldots$.

LEMMA 2. *If the prime factors of the positive integer $b > 1$ are $p_1, p_2, \ldots, p_m$, then*

$$\sum_{d \mid b} 1/d < 1 \Big/ \prod_{i=1}^{m}(1 - 1/p_i).$$

Lemma 1 follows from the result $p_{i-1} \leqslant p_i - 1, i \geqslant 2$, and Lemma 2 follows from the result

$$\sum_{d \mid b} 1/d < \prod_{i=1}^{m}\left(\sum_{k=0}^{\infty}(1/p_i)^k\right).$$

Using these two lemmas we now proceed to settle the conjecture by induction. Since $f(1) = f(2) = f(3) = 1$, the conjecture holds for the first few positive integers. Suppose the conjecture holds for all positive integers less than n. If $n = q_1^{a_1} q_2^{a_2} \cdots q_m^{a_m}, a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$, is the prime factorization of n and $c = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, then $f(n) = f(c)$ and $c \leqslant n$. Thus if $c < n$, then $f(n) = f(c) \leqslant c < n$ by the induction hypothesis and we would be done.

There remains the case $n = c$. We further subdivide this case into the subcases $a_m = 1$ and $a_m > 1$. If $m = a_m = 1$, then $n = 2$, a value of n for which we know the conjecture to be true. If $a_m = 1$ and $m \geqslant 2$, then every factorization of n has precisely one factor divisible by p_m so that, if $b = n/p_m$, we have

$$f(n) = \sum_{d \mid b} f(b/d) \leqslant b \sum_{d \mid b} 1/d < b \Big/ \prod_{i=1}^{m-1}(1 - 1/p_i) \leqslant b \cdot p_{m-1} < n,$$

by the induction hypothesis and the two lemmas.

Finally, if $a_m > 1$, then set $e = (n/p_m)p_{m+1}$. For any factorization of n, say $n = d_1 d_2 \cdots d_s$, where d_1 is a largest factor divisible by p_m, then $((d_1/p_m)p_{m+1})d_2 d_3 \cdots d_s = e$. Thus essentially different factorizations of n yield in this manner essentially different factorizations of e, so we have $f(n) \leqslant f(e)$. Similar to before, if $b = e/p_{m+1}$, we have

$$f(n) \leqslant f(e) = \sum_{d \mid b} f(b/d) \leqslant b \sum_{d \mid b} 1/d < b \Big/ \prod_{i=1}^{m}(1 - 1/p_i) \leqslant b \cdot p_m = n,$$

and the induction is complete.

Reference

1. J. F. Hughes and J. O. Shallit, On the number of multiplicative partitions, this MONTHLY, 90 (1983) 468–471.

GEOMETRIC SERIES AND A PROBABILITY PROBLEM

CURTIS COOPER

Department of Mathematics and Computer Science, Central Missouri State University, Warrensburg, MO 64093

Suppose

$$a_1 + a_2 + a_3 + \cdots$$

is a geometric series with ratio r, $|r| < 1$. We know that the sum of this series is

$$\frac{a_1}{1 - r}.$$

Now if n and $k_1, k_2, \ldots, k_n$ are positive integers, what is $\sum_{i_1 < i_2 < \cdots < i_n} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_n}^{k_n}$? In this note we will answer this question and then use the result to solve a probability problem.

THEOREM. *Let*

$$a_1 + a_2 + a_3 + \cdots$$

be a geometric series with ratio r, $|r| < 1$. *Let* n *and* $k_1, k_2, \ldots, k_n$ *be positive integers. Then*

$$\sum_{i_1 < i_2 < \cdots < i_n} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_n}^{k_n} = a_1^{k_1+k_2+\cdots+k_n} r^{k_2+2k_3+\cdots+(n-1)k_n} \frac{1}{1 - r^{k_1+\cdots+k_n}} \times \frac{1}{1 - r^{k_2+\cdots+k_n}} \cdots \frac{1}{1 - r^{k_n}}.$$

Proof. By induction on n.

$$a_1^k + a_2^k + a_3^k + \cdots = \frac{a_1^k}{1 - r^k},$$

so the result is true for $n = 1$.

Assume the result is true for $n = m$. Then

$$\begin{aligned}
\sum_{i_1 < i_2 < \cdots < i_{m+1}} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_{m+1}}^{k_{m+1}} &= \sum_{i_1 < i_2 < \cdots < i_m} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_m}^{k_m} \left(a_{i_m+1}^{k_{m+1}} + a_{i_m+2}^{k_{m+1}} + \cdots \right) \\
&= \sum_{i_1 < i_2 < \cdots < i_m} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_m}^{k_m} a_{i_m+1}^{k_{m+1}} (1 + r^{k_{m+1}} + \cdots) \\
&= \sum_{i_1 < i_2 < \cdots < i_m} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_m}^{k_m} (r a_{i_m})^{k_{m+1}} (1 + r^{k_{m+1}} + \cdots) \\
&= \frac{r^{k_{m+1}}}{1 - r^{k_{m+1}}} \sum_{i_1 < i_2 < \cdots < i_m} a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_m}^{k_m + k_{m+1}} \\
&= a_1^{k_1+k_2+\cdots+k_{m+1}} r^{k_2+2k_3+\cdots+mk_{m+1}} \frac{1}{1 - r^{k_1+\cdots+k_{m+1}}} \\
&\quad \times \frac{1}{1 - r^{k_2+\cdots+k_{m+1}}} \cdots \frac{1}{1 - r^{k_{m+1}}}.
\end{aligned}$$

Thus, the theorem is true by induction.

We now use this theorem to solve the following problem:

Three players, A, B, and C, take turns throwing a single die, A leads. As soon as a player tosses a one, that player drops out of the game and the remaining players continue rolling the die until everyone has rolled a one. What is the probability that A tosses the first one, B tosses the second one, and C tosses the third one?

To solve this problem, let

$$\frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 + \cdots$$

be the geometric series $a_1 + a_2 + a_3 + \cdots$, where a_i is the probability that player A (,B, or C) throws a one for the first time on the ith toss. The probability that the game ends after A's ith toss, B's jth toss, and C's kth toss is $a_i a_j a_k$. In addition, A will roll the first one, B will roll the second one, and C will roll the third one if and only if $i < j < k$, $i < j = k$, $i = j < k$, or $i = j = k$. Therefore, applying the theorem, the probability that the order of finish is A, B, and C is

$$\sum_{i<j<k} a_i a_j a_k + \sum_{i<j} a_i a_j^2 + \sum_{i<k} a_i^2 a_k + \sum_i a_i^3 = \frac{125}{1001} + \frac{25}{1001} + \frac{5}{91} + \frac{1}{91} = \frac{216}{1001}.$$