On the Natural Density of the Niven Numbers

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Properties of the digits of integers have always intrigued mathematicians. In particular, digital sums of the integers have been the subject of much study. Our objective is to demonstrate a fascinating property of Niven numbers—numbers so named in honor of Ivan Niven, who sparked the investigation of these integers at a conference devoted to number theory [3].

Definition. An integer is called a *Niven number* if it is divisible by its digital sum.

Some examples of Niven numbers are 8, 12, 180, and 4050. The set of Niven numbers is infinite since any positive integral power of 10 is a Niven number.

Even though a variety of ideas, results and open questions were considered in [1], [2] and in papers presented at various mathematics meetings, the natural density of the set of Niven numbers has been unanswered until now. Let $N(x)$ denote the number of Niven numbers not exceeding x. We shall show that the natural density of the Niven numbers is zero; that is,

$$\lim_{x\to\infty} \frac{N(x)}{x} = 0.$$

To investigate the natural density of the set of Niven numbers, we let $S(n)$ denote the digital sum of n and establish the following interesting result:
The mean μ and the standard deviation σ of $\{S(0), S(1), S(2), \ldots, S(10^n - 1)\}$ are

$$\mu = (4.5)n \quad \text{and} \quad \sigma = \sqrt{(8.25)n}\,. \tag{1}$$

This can be justified by considering a random experiment consisting of throwing n ten-faced dice, where each of the ten faces is marked with one of the numbers $0, 1, 2, \ldots, 9$. The sample space associated with this experiment consists of 10^n points

$$\{(x_1, x_2, \ldots, x_n) \mid 0 \leqslant x_i \leqslant 9\}.$$

Each outcome represents the digits of a number in the interval $0 \leqslant x < 10^n$. Hence,

$$\frac{1}{10^n} \sum_{x=0}^{10^n - 1} S(x) = \mu = E(x_1 + x_2 + \cdots + x_n) = nE(x_1)$$

and

$$\frac{1}{10^n} \sum_{x=0}^{10^n - 1} [S(x) - \mu]^2 = \sigma^2 = \mathrm{Var}(x_1 + x_2 + \cdots + x_n) = n\,\mathrm{Var}(x_1).$$

But

$$E(x_1) = \frac{1}{10}(0 + 1 + 2 + \cdots + 9) = 4.5$$

and

$$\begin{aligned} \mathrm{Var}(x_1) &= E(x_1^2) - [E(x_1)]^2 \\ &= \frac{1}{10}[0^2 + 1^2 + 2^2 + \cdots + 9^2] - (4.5)^2 = 8.25. \end{aligned}$$

Therefore, $\mu = (4.5)n$ and $\sigma^2 = (8.25)n$.

For $k \geqslant 1$, we now define the following sets of integers:

$$\begin{aligned} I &= \{x \in [0, 9n] : |x - \mu| \leqslant k\sigma\} \\ A_1 &= \{x \in [0, 10^n) : x \text{ is a multiple of a member of } I\} \\ A_2 &= \{x \in [0, 10^n) : |S(x) - \mu| \geqslant k\sigma\} \\ N &= \{x \in [0, 10^n) : x \text{ is a Niven number}\}. \end{aligned}$$

Using these sets, we shall prove that the density of the Niven numbers is zero; that is,

$$\lim_{x \to \infty} \frac{N(x)}{x} = 0.$$

For any positive integer n, we have

$$N \subseteq A_1 \cup A_2. \qquad (2)$$

This can be seen as follows: If $x \in N$, then x is a multiple of $S(x)$ and $S(x) \in [0,9n] = I \cup ([0,9n] - I)$. Thus, $S(x) \in I$ or $S(x) \in [0,9n] - I$. In the first case, $x \in A_1$ whereas in the second case $x \in A_2$. Therefore, $x \in A_1 \cup A_2$.

It follows from (2) that

$$N(10^n) \leqslant |A_1| + |A_2|,$$

where $|A_i|$ denotes the number of elements of A_i $(i = 1, 2)$. Since $\left[\frac{10^n}{t}\right]$ is the number of nonzero multiples of t not exceeding 10^n, we have

$$|A_1| \leqslant \sum_{t \in I} \left[\frac{10^n}{t} \right].$$

Here, as usual, the square brackets indicate the integral-part operator.

Let us now recall Chebyshev's Inequality ([4] page 55, Theorem 7), which states that: if X is a random variable with standard deviation σ and mean μ, then

$$\Pr(|X - \mu| \geqslant k\sigma) \leqslant \frac{1}{k^2}$$

for every $k > 0$. Hence, it follows that

$$\frac{|A_2|}{10^n} \leqslant \frac{1}{k^2} .$$

Therefore,

$$N(10^n) \leqslant \sum_{t \in I} \left[\frac{10^n}{t} \right] + \frac{10^n}{k^2} \leqslant 10^n \left(\sum_{t \in I} \frac{1}{t} + \frac{1}{k^2} \right). \tag{3}$$

Noting that (3) holds for all n and any fixed $k \geqslant 1$, we may take $k = n^{1/4}$. Then, since

$$\sum_{t \in I} \frac{1}{t} \leqslant \int_{\mu - k\sigma}^{\mu + k\sigma} \frac{dt}{t} + \frac{1}{\mu - k\sigma} = \ln\left(\frac{\mu + k\sigma}{\mu - k\sigma} \right) + \frac{1}{\mu - k\sigma}$$

(the integral exists since $\mu - k\sigma = (4.5)n - (n^{1/4})(8.25n)^{1/2} > 0$), it follows that

$$\frac{N(10^n)}{10^n} \leqslant \ln\left(\frac{\mu + k\sigma}{\mu - k\sigma} \right) + \frac{1}{\mu - k\sigma} + \frac{1}{k^2} .$$

Hence,

$$\frac{N(10^n)}{10^n} \leqslant \ln\left[\frac{(4.5)n + (n^{1/4})(8.25n)^{1/2}}{(4.5)n - (n^{1/4})(8.25n)^{1/2}} \right] + \frac{1}{(4.5)n - (n^{1/4})(8.25n)^{1/2}} + \frac{1}{(n^{1/4})^2} ,$$

and so $\lim_{n \to \infty} \dfrac{N(10^n)}{10^n} = 0$. Since for any $x \geqslant 1$, there exists a natural number n such that

$$10^{n-1} \leqslant x < 10^n ,$$

we have

$$\frac{N(x)}{x} \leqslant \frac{N(10^n)}{10^{n-1}} = 10\left(\frac{N(10^n)}{10^n} \right).$$

Therefore, $\lim_{x \to \infty} \dfrac{N(x)}{x} = 0$.

Within the last five years, our feelings about the question of the natural density of the Niven numbers varied from conjecturing that the density exists and is nonzero, to conjecturing that it may not exist, to conjecturing that it exists and is zero. [Readers who would like to investigate the natural density of other special sequences might wish to consult ([5], chapter 11).] As the above discussion demonstrates, the latter conjecture is valid. The proof of this final conjecture was motivated by a conversation with Carl Pomerance at the Regional AMS meeting held in Austin, Texas in 1981. Some of the most crucial ideas in this proof emanated from that discussion. We would like to thank him here for his help.

Acknowledgments. The authors wish to thank the referees for their useful and helpful comments in general and in particular the statistical proof of the lemma.

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The Author and the Editors

R. P. Boas

I send the paper in:
They say it is too thin.
When I've corrected that
They say it is too fat.

The next word that I hear:
"It isn't really clear."
But after I explain
They look at it again.

At last they write to say
There's been so much delay
That with regret they find
(And hope I will not mind)—
Admittedly, it's sad—
But someone else just published everything I had!