Bach, 5465, and Upside-Down Numbers

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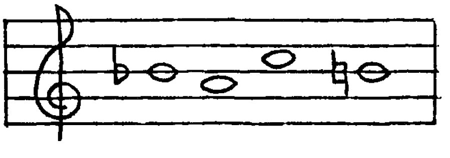
Music with unusual characteristics has been of interest to composers of all musical periods. For example, the theme



(1)

has been used by J. S. Bach in *The Art of the Fugue*, Robert Schumann in *Sechs Fugen uber Bach*, and in *Fantasy and Fugue on B.A.C.H.* by Franz Liszt. Note that the German names of the musical notes of (1) spell B (B flat), A, C, and H (B natural). Without accidentals, this motif sounds the same when it is played right-side up or upside down. Thus, we have what could be called "upside-down" music. In Paul Hindemith's *Ludus Tonalis*, the postlude can be played by turning the pages of the prelude upside-down and vice versa, when allowance is made for accidentals.

Thinking about the structure of such music led us to investigate what may be called "upside-down" numbers. Although any base may be used, we will be concerned only with base-ten integers. It can be shown that musically, this would mean that we are concerned only with musical strings of the nine whole notes, which are completely in the musical staff.

Upside-Down Numbers. We call a positive integer an "upside-down" number if its ith leftmost digit and its ith rightmost digit are complements—that is, their sum is 10. Thus, 5465 is upside down since $5+5=4+6=10$. The definition of an upside-down number implies that it has no zero digit. Note also that the middle digit must be 5 for an upside-down number having an odd number of digits.

For any positive integer m, we denote by $\overleftarrow{m}$ the integer formed by reversing the digits of m. Thus (by the definition) m is upside down if and only if

$$m + \overleftarrow{m} = \sum_{k=1}^{d} 10^k,$$

where d is the number of digits of m. Since $10^{d-1} \leq m < 10^d$,

$$d = [\log m] + 1, \tag{2}$$

where (the base of the logarithm is 10) the square brackets denote the integral part operator.

The following partial list of upside-down numbers may be helpful for obtaining a better grasp of their properties. Observe that the nth column can be generated using the numbers in the $(n-2)$nd and the 2nd columns.

$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
5	19	159	1199	11599
	28	258	1289	12589
	37	357	·	·
	46	456	·	·
	55	555	·	·
	64	654	1919	19519
	73	753	2198	21598
	82	852	2288	22588
	91	951	·	·
			·	·
			·	·
			5465	54565
			·	·
			·	·
			·	·
			9911	99511

There are other ways to construct upside-down numbers. In fact, for any positive integer having no zero digit, one can easily construct an upside-down number using the given integer as its initial digits. For example, consider the integer 54. The upside-down number 5465, using 54 as the initial two digits, is obtained by annexing the reversal of the two-digit number formed by subtracting each digit of 54 from 10.

How many upside-down numbers not exceeding x are there?

To consider this and other related questions, we henceforth let $U(x)$ denote the number of upside-down numbers not exceeding the real number x.

The Order of Magnitude of $U(x)$. Let k be a positive integer greater than or equal to 3. By our observation preceding the list of upside-down numbers, we have

$$U(10^k) - U(10^{k-1}) = 9\big(U(10^{k-2}) - U(10^{k-3})\big). \tag{3}$$

Using (3) and mathematical induction, one can readily establish that

$$U(10^n) - U(10^{n-1}) = 9^{[n/2]} \tag{4}$$

for every positive integer n. Summing each side of (4), from $n=1$ to $n=m$, we obtain the exact value

$$U(10^m) = \sum_{n=1}^{m} 9^{[n/2]} \tag{5}$$

for each positive integer m.

To determine the asymptotic behavior of $U(x)$, first observe that (5) and (4) yield

$$U(x) \leq \sum_{k=1}^{[\log x]+1} 9^{[k/2]} \leq \sum_{k=1}^{[\log x]+1} 3^k = \frac{3^{[\log x]+2} - 3}{2} \tag{6}$$

for each real number $x > 0$. Since

$$[\log x] \leq \log x \quad \text{and} \quad 3^{\log x} = x^{\log 3},$$

we see that $U(x)/x^{\log 3}$ is bounded. This is frequently expressed in "big oh" notation:

$$U(x) = O(x^{\log 3}). \tag{7}$$

From (5), we also have

$$\frac{U(10^{2n})}{(10^{2n})^{\log 3}} = \frac{U(10^{2n})}{3^{2n}} = \frac{1 + 2\left(\sum_{i=1}^{n-1} 3^{2i}\right) + 3^{2n}}{3^{2n}}.$$

Therefore,

$$\lim_{n\to\infty} \frac{U(10^{2n})}{(10^{2n})^{\log 3}} = \lim_{n\to\infty} \frac{(-5/4) + (5/4)(3^{2n})}{3^{2n}} = \frac{5}{4}. \tag{8}$$

Similarly,

$$\lim_{n\to\infty} \frac{U(10^{2n+1})}{(10^{2n+1})^{\log 3}} = \lim_{n\to\infty} \frac{(-5/4) + (3/4)(3^{2n+1})}{3^{2n+1}} = \frac{3}{4}. \tag{9}$$

Thus, (7) is confirmed as the correct order of magnitude for $U(x)$. However, (8) and (9) show that $U(x)$ is not asymptotic to any constant multiple of $x^{\log 3}$. That is, the ratio $U(x)/x^{\log 3}$ does not approach a fixed constant as x increases without bound.

It is possible to determine bounds for $U(x)/x^{\log 3}$. In fact, let us show that for any $x > 0$:

$$\frac{16}{81} \leq \frac{U(x)}{x^{\log 3}} < \frac{15}{4}. \tag{10}$$

We begin by determining n such that

$$10^{n-1} \leq x < 10^n.$$

Then $n - 1 \leq \log x < n$, and (since $3^{\log x} = x^{\log 3}$)

$$\frac{U(10^{n-1})}{3^n} \leq \frac{U(x)}{x^{\log 3}} < \frac{U(10^n)}{3^{n-1}},$$

which can be written as

$$\frac{1}{3}\left(\frac{U(10^{n-1})}{3^{n-1}}\right) \leq \frac{U(x)}{x^{\log 3}} < 3\left(\frac{U(10^n)}{3^n}\right). \tag{11}$$

Since, by (8) and (9), the respective maximum and minimum of $U(10^k)/3^k$ is $\frac{5}{4}$ and $\frac{16}{27}$, our bounds in (10) follow.

Remark. In terms of how spread out upside-down numbers are, (10) demonstrates that the natural density of upside-down numbers is zero. Specifically,

$$\lim_{x \to \infty} \frac{U(x)}{x} = 0.$$

Suppose u_n denotes the nth upside-down number. In the next section, we will give bounds for u_n as well as an algorithm for determining u_n for any given n.

Bounds and an Algorithm for u_n. Letting $x = u_n$ in (10), we have

$$\left(\frac{16}{81}\right) u_n^{\log 3} \le U(u_n) < \left(\frac{15}{4}\right) u_n^{\log 3}.$$

Since $U(u_n) = n$, the above inequalities can be recast as

$$\left(\frac{4n}{15}\right)^{1/\log 3} < u_n \le \left(\frac{81n}{16}\right)^{1/\log 3}. \tag{12}$$

Noting that $n^{1/\log 3} = n^{\log_3 10}$ and $n^{1/\log 3} \approx n^2$, we see that

$$u_n = O(n^{\log_3 10}) \tag{13}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{u_n} \text{ converges.}$$

To find the nth upside-down number, we use (4) to determine the positive integer d such that $U(10^{d-1}) < n \le U(10^d)$. The following algorithm illustrates how to find u_n for any given n.

Step 1. Find d such that $U(10^{d-1}) < n \le U(10^d)$.

Step 2. Let $m = n - U(10^{d-1})$.

Step 3. Change m to base 9, and subtract 1 using base 9 arithmetic.

Step 4. Consider the result of Step 3 as a base 10 number and add $(10^{[d/2]} - 1)/9$. Call this result L.

Step 5. If d is even, the nth upside-down number is found by annexing L with the complement of L. If d is odd, the nth upside-down number is found by annexing L with 5 annexed with the complement of L.

Example. Let us find u_{59}. Since $U(10^3) = 19$ and $U(10^4) = 100$, we have that $d = 4$. Thus, the value of m in Step 2 is 40. Since m in base 9 is 44, the result of Step 3 is 43. Thus, $L = 54$ by Step 4. From this and the fact that d is even, it follows, by Step 5, that the 59th upside-down number is 5465. (Note that the digits of 5465, respectively, represent the levels of the staff that the musical notes B, A, C, and H lie on.)

To clarify this algorithm, note that Step 1 calculates the number d of digits of u_n. Step 2 locates u_n in the interval $(10^{d-1}, 10^d)$. To explain Steps 3 and 4, consider the following columns from our generated list of upside-down numbers:

$d=2j$	$d=2j+1$
111...19...999	111...159...999
111...28...999	111...258...999
⋮	⋮
111...91...999	111...951...999
⋮	⋮
555...55...555	555...555...555
⋮	⋮
999...91...111	999...951...111

Note that the leftmost halves of d-digit upside-down numbers constitute the set of numbers, having no zero digit, from

$$(10^{[d/2]}-1)/9 \quad \text{to} \quad 10^{[d/2]}-1. \tag{14}$$

So, after subtracting one from each digit of each integer in (14), we have all the base 9 numbers from

$$0 \quad \text{to} \quad (8)(10^{[d/2]}-1)/9. \tag{15}$$

Therefore, Steps 3 and 4 construct L, the leftmost half of u_n, by finding the mth number in (14). Finally, Step 5 tells what digits to annex to L to determine u_n.

Conclusion. Although the algorithm given above determines u_n for any given positive integer n, a formula for u_n has not been found by the authors. In view of (8) and (9), the authors feel that no asymptotic formula exists for $U(x)$.

Just as "upside-down" music has interesting musical aspects, we see that the mathematical analogue—upside-down numbers—also yields mathematical questions.

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> Mathematics is on the artistic side a creation of new rhythms, orders, designs, harmonies, and on the knowledge side, is a systematic study of various rhythms, orders, designs and harmonies.
>
> William L. Schaaff