The same figure can be used to show that $\frac{d}{dx}(\cos x) = -\sin x$. Using $GC = \cos x - \cos(x + \Delta x)$, it follows from right triangle DGC that chord $DC = \dfrac{\cos x - \cos(x + \Delta x)}{\sin\left(x + \frac{\Delta x}{2}\right)}$. Again, using (*), we get

$$\Delta x > \frac{\cos x - \cos(x + \Delta x)}{\sin\left(x + \frac{\Delta x}{2}\right)} > \frac{\cos(x + \Delta x)}{\cos x} \cdot \Delta x$$

or

$$-\sin\left(x + \frac{\Delta x}{2}\right) < \frac{\cos(x + \Delta x) - \cos x}{\Delta x} < \frac{\cos(x + \Delta x)}{\cos x}\left(-\sin\left(x + \frac{\Delta x}{2}\right)\right).$$

Thus,

$$\frac{d}{dx}(\cos x) = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = -\sin x.$$

It should be remarked that, although the angles were restricted to the first quadrant, the chain rule can be used to readily extend these results to other quadrants. Finally, we observe that the formulas $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ are immediate consequences of our formulas, since

$$\lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(x + h) - \cos x}{h} = -\sin x$$

for $x = 0$.

Application of a Generalized Fibonacci Sequence

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In the November 1979 Classroom Capsules Column, Michael Chamberlain gave a solution to the following problem:

A fair coin is tossed repeatedly until n consecutive heads are obtained. What is the expected number of tosses e_n to conclude the experiment? (*)

This capsule offers a nice illustration of how a generalized Fibonacci sequence can be used to solve the above expectation problem.

Given the positive integer n in (*), let

$$f_i = \begin{cases} 0, & i = 1, 2, \ldots, n-1 \\ 1, & i = n \\ \sum_{k=1}^{n} f_{i-k}, & i > n. \end{cases} \tag{1}$$

Since each term f_{n+k} $(k > 0)$ in (1) is the sum of the preceding n terms,

$$f_{n+k} = f_{n+k-1} + f_{n+k-2} + \cdots + f_k, \tag{2}$$

we see that this generalized Fibonacci sequence reduces to the usual Fibonacci sequence when $n = 2$.

Now let $S = \{H^n; TH^n; T^2H^n, HTH^n; T^3H^n, THTH^n, HT^2H^n, H^2TH^n; \ldots\}$ be the sample space for successful experiments in (*). Then f_i is the number of elements in S consisting of exactly i flips. To verify that

$$P(S) = \sum_{i=1}^{\infty} f_i \cdot \frac{1}{2^i}$$

equals 1 and to evaluate

$$e_n = \sum_{i=1}^{\infty} i \cdot f_i \cdot \frac{1}{2^i}, \tag{3}$$

we shall show that

$$B(x) \equiv \sum_{i=1}^{\infty} f_i \cdot x^i = \frac{x^n}{1 - x - x^2 - \cdots - x^n}. \tag{4}$$

Recall that if $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ over some interval I, the product $C(x) = A(x) \cdot B(x)$ can be obtained by termwise multiplication of the power series for $A(x)$ and $B(x)$. Collecting terms with equal powers of x, we find that the coefficient c_k of x^k in $C(x) = \sum_{k=0}^{\infty} c_k x^k$ $(x \in I)$ is given by

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0. \tag{5}$$

Since $f_{n+k+1} = 2f_{n+k} - f_k$ from (2) and since $f_k > 0$ for $k > n - 1$, we know (the ratio test) that $\sum_{k=0}^{\infty} f_k x^k$ converges to some function $B(x)$ for $|x| \leqslant \frac{1}{2}$. Taking

$$a_k = \begin{cases} 1, & k = 0 \\ -1, & 1 \leqslant k \leqslant n \\ 0, & k > n \end{cases} \quad \text{and} \quad b_k = f_k$$

and using (2), we see that (5) yields $c_n = 1$ and $c_k = 0$ for $k \neq n$. Therefore, $A(x) = 1 - x - x^2 - \cdots - x^n$ and $C(x) = x^n$ yield $B(x)$ as in (4).

Using $B(x)$ and its derivative $B'(x)$, we find immediately that $P(S) = B(\frac{1}{2}) = 1$, and $e_n = \frac{1}{2} B'(\frac{1}{2})$. Since $B'(\frac{1}{2}) = 2^{n+2} - 4$, the answer sought in (*) is $e_n = 2^{n+1} - 2$.

Editor's Note: $B(x)$ is the generating function for the generalized Fibonacci sequence (1). For a further discussion of generating functions in probability theory, see W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, 3rd ed., Wiley, New York, 1968.

Queries

9. Can anyone provide an example of an elementary transcendental function having three different kinds of asymptotes ($x - a = 0$, $y - b = 0$, $y = mx + c$) and defined by one single equation in its domain of definition?

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