A Generating Function for the Distribution of the Scores of all Possible Bowling Games

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1. Introduction Assuming that you know the terminology, rules of play, and the method of scoring, we will determine the number of ways that any particular score can occur in an ordinary game of ten-pin bowling. For example, it is clear that there is exactly one way that a perfect score of 300 can be obtained (all strikes) and exactly one way that a score of 0 can be obtained (all gutter balls). It may not be easy to see, however, that there are exactly 50613244155051856 ways of obtaining a score of 100.

Consider the following "line" of a bowling game:

5 4	9 /	X	X	X	7 1	X	6 /	4 2	3 / X
9	29	59	86	104	112	132	146	152	172

(1)

The lower number in each frame is the cumulative score. The upper numbers in each frame are the number of pins knocked down by the first and second ball respectively. The "/" indicates that the remaining pins were knocked down by the second ball and is called a "spare." The "X" indicates that all ten pins were knocked down by the first ball and is called a "strike."

Note that the game given in (1) can be represented by the sequence of nine ordered pairs and an ordered triple,

$$(5,4),(9,1),(10,0),(10,0),(10,0),(7,1),(10,0),(6,4),(4,2),(3,7,10) \quad (2)$$

With (2) as a model, we define a "bowling game" as a sequence of the form

$$(x_1,y_1),(x_2,y_2),\ldots,(x_9,y_9),(x_{10},y_{10},z_{10}), \quad (3)$$

where the terms of (3) represent the ten frames of a game and each component of a term denotes the number of pins knocked down by that ball. Here, x_i, y_i and z_{10} are nonnegative integers where

$$x_i + y_i \leqslant 10 \qquad \text{for} \quad i = 1,\ldots,9, \quad (4)$$

and with somewhat more involved conditions given in (5) on x_{10}, y_{10}, and z_{10}.

It was shown in [1] that the number of all possible bowling games is $(66^9)(241)$ which is approximately 5.7 billion billion and that the mean of all possible bowling games is approximately 80.

Thus, to determine the exact distribution of all possible bowling games by a computer generation of all possibilities would require in excess of 180 years even if every computer operation would take less than one-billionth of a second to perform. We will avoid this problem by constructing a generating function which determines the distribution of the scores of all possible bowling games. To do this, we will use the following sets where all components are nonnegative integers:

$$
\begin{aligned}
A &= \{(x,y) : x+y \leqslant 9\} \\
B &= \{(x,y,0) : (x,y) \in A\} \\
&\quad \cup \{(x, 10-x, z) : x \leqslant 9; z \leqslant 10\} \\
&\quad \cup \{(10, y, z) : y \leqslant 9; y+z \leqslant 10\} \\
&\quad \cup \{(10,10,z) : z \leqslant 10\}.
\end{aligned} \tag{5}
$$

Therefore, A is the set of frames in which a "mark" (spare or strike) is not made, and the set B is the set of all possibilities for the tenth frame. We wish to determine a polynomial function of the form

$$P(t) = \sum_{i=0}^{300} s_i t^i, \tag{6}$$

where s_i is the number of ways that a score of i is made. For example, $s_0 = 1$, $s_{300} = 1$, and $s_{100} = 50613244155051856$. Thus, $P(t)$ is a generating function for the set of all possible bowling scores.

2. States To aid in finding the function given in (6), we define four "states" in the process of calculating the score of a game. They are called the "OPEN state", the "SPARE state", the "STRIKE state", and the "DOUBLE state." These are defined as follows:

1. The OPEN state describes that the current frame is open. (7)
2. The SPARE state describes that a spare has been made in the current frame.
3. The STRIKE state describes that a strike has been made in the current frame and that either an open or spare was made in the previous frame.
4. The DOUBLE state describes that a strike was made in both the current and previous frames.

We observe here that all capital letters are used in reference to a state so as to emphasize that the state of a frame is not the same as what was rolled in that frame. For example, a frame in which a strike is rolled is not necessarily in the STRIKE state since the previous frame may not have been an open or a spare.

As we bowl a game and keep score, we pass from one state to another state. By convention, we will assume that each game starts with a 0th frame which is in the OPEN state with an accumulated score of 0. To clarify the above terminology, consider the bowling line given in (1). The 0th, 1st, 6th, and 9th frames are in the OPEN state, the 2nd and 8th frames are in the SPARE state, the 3rd and 7th frames are in the STRIKE state, while the 4th and 5th frames are in the DOUBLE state. Since the current state of a frame will determine the contribution of the next frame to the accumulated score, we do not need to define a state for the 10th frame.

3. Generating functions of transitions Here, we will determine the generating functions for the 16 possible transitions from one state to another state. First, it is clear that the transitions

$$
\begin{array}{lcl}
\text{OPEN} & \text{to} & \text{DOUBLE}, \\
\text{SPARE} & \text{to} & \text{DOUBLE}, \\
\text{STRIKE} & \text{to} & \text{STRIKE},
\end{array} \tag{8}
$$

and

$$\text{DOUBLE} \quad \text{to} \quad \text{STRIKE}$$

cannot occur and hence may be considered as having a generating function of 0. To determine the generating function of a transition where (x, y) is the second state, we define the "value" of this transition as the contribution of the second state to the accumulated score. The following list gives the value of each of the other 12 transitions in terms of x and y.

$$
\begin{array}{llll}
\text{OPEN} & \text{to OPEN} & \text{has a value of} & x+y \\
\text{OPEN} & \text{to SPARE} & \text{has a value of} & 10 \\
\text{OPEN} & \text{to STRIKE} & \text{has a value of} & 10 \\
\text{SPARE} & \text{to OPEN} & \text{has a value of} & 2x+y \\
\text{SPARE} & \text{to SPARE} & \text{has a value of} & x+10 \\
\text{SPARE} & \text{to STRIKE} & \text{has a value of} & 20 \\
\text{STRIKE} & \text{to OPEN} & \text{has a value of} & 2x+2y \\
\text{STRIKE} & \text{to SPARE} & \text{has a value of} & 20 \\
\text{STRIKE} & \text{to DOUBLE} & \text{has a value of} & 20 \\
\text{DOUBLE} & \text{to OPEN} & \text{has a value of} & 3x+2y \\
\text{DOUBLE} & \text{to SPARE} & \text{has a value of} & x+20 \\
\text{DOUBLE} & \text{to DOUBLE} & \text{has a value of} & 30.
\end{array}
\tag{9}
$$

Thus, considering the information in (8) and (9), we see that the generating functions for the 16 transitions are given by the following transition matrix, T.

$$
\begin{pmatrix}
\sum\limits_{(x,y)\in A} t^{x+y} & 10t^{10} & t^{10} & 0 \\
\sum\limits_{(x,y)\in A} t^{2x+y} & \sum\limits_{x=0}^{9} t^{x+10} & t^{20} & 0 \\
\sum\limits_{(x,y)\in A} t^{2x+2y} & 10t^{20} & 0 & t^{20} \\
\sum\limits_{(x,y)\in A} t^{3x+2y} & \sum\limits_{x=0}^{9} t^{x+20} & 0 & t^{30}
\end{pmatrix}
\tag{10}
$$

The rows of the matrix T represent the first state while the columns of T represent the second state in the order OPEN, SPARE, STRIKE, and DOUBLE. For example, the entry in the third row and second column of T is the generating function for the transition STRIKE to SPARE.

In addition, the column matrix

$$
C = \begin{pmatrix}
\sum\limits_{(x,y,z)\in B} t^{x+y+z} \\
\sum\limits_{(x,y,z)\in B} t^{2x+y+z} \\
\sum\limits_{(x,y,z)\in B} t^{2x+2y+z} \\
\sum\limits_{(x,y,z)\in B} t^{3x+2y+z}
\end{pmatrix}
\tag{11}
$$

represents the contribution made by the 10th frame depending on whether the 9th frame is in the OPEN, SPARE, STRIKE, or DOUBLE state.

Since T^9 will be the matrix that gives the generating functions for all possible scores commencing with a given state and terminating with another state through nine transitions, it follows that the generating function $P(t)$ in (6) will be the entry in the one-by-one matrix

$$RT^9C \tag{12}$$

where $R=(1,0,0,0)$.

Fortunately, we do not have to actually calculate and simplify the matrix expression in (12). An Apple IIe Pascal program was written which uses (12) and determines the coefficient of each term of (6) and hence finds the exact number of ways that each bowling score can occur. This program is available upon request. The distribution of all possible bowling scores generated by this program is listed in appendix A.

APPENDIX A

DISTRIBUTION OF BOWLING SCORES

0	1	50	11193770355829009	100	50613244155051856
1	20	51	13810930667765157	101	45887089510794122
2	210	52	16878453276117746	102	41483436078768079
3	1540	53	20435326129713654	103	37397371704961189
4	8855	54	24515635362932954	104	33621048067136846
5	42504	55	29146610869639549	105	30144388614623696
6	177100	56	34346628376654913	106	26955619314626157
7	657800	57	40123251227815383	107	24041709119775647
8	2220075	58	46471404549689351	108	21388640692533960
9	6906900	59	53371780703441318	109	18981680119465910
10	20030010	60	60789577452586487	110	16805547548715206
11	54627084	61	68673668434334934	111	14844654231857239
12	141116637	62	76956298564663402	112	13083276623221517
13	347336412	63	85553384395717227	113	11505812292077067
14	818558424	64	94365480254213528	114	10096971927616045
15	1854631380	65	103279445170253902	115	8842020009154293
16	4053948342	66	112170812747354087	116	7726929590817265
17	8574134256	67	120906827121834566	117	6738528470417086
18	17590903116	68	129350064451661348	118	5864552560171552
19	35084425512	69	137362512979745598	119	5093653838062639
20	68153183370	70	144809940796620325	120	4415377510495980
21	129156542039	71	151566341291631624	121	3820097597373727
22	239128282128	72	157518221668013078	122	3298981687014508
23	433093980298	73	162568486673578693	123	2843905747206868
24	768175029950	74	166639683923175378	124	2447444695948898
25	1335679056261	75	169676402232105648	125	2102793053565659
26	2278764308864	76	171646676234883305	126	1803790254604935
27	3817721269708	77	172542309343731946	127	1544848145184291
28	6285424931278	78	172378125687965848	128	1320992367181792
29	10176048813473	79	171190226627438257	129	1127775864826813
30	16210652213304	80	169033430825208027	130	961294388171457
31	25423690787719	81	165978103316094584	131	818085023387881
32	39274771758064	82	162106654714921075	132	695128788327698
33	59789973730461	83	157509948809043576	133	589753122859383
34	89736657900900	84	152283892386077931	134	499630252931260
35	132834787033075	85	146526364181517039	135	422696870992462
36	194006223597572	86	140334651650668803	136	357151976811922
37	279661205716974	87	133803399444707801	137	301400973036441
38	398018151390200	88	127023103852577896	138	254052574077937
39	559449136091831	89	120079021507938035	139	213889601295347
40	776838931567572	90	113050455155943519	140	179862464456172
41	1065940588576732	91	106010240661754449	141	151065169242834
42	1445705502357343	92	99024411737621323	142	126722015973414
43	1938561121705315	93	92151904402003308	143	106169469752641
44	2570605432880903	94	85444345654857875	144	88840622360686
45	3371684590465908	95	78945863453573001	145	74252067274687
46	4375319099346208	96	72693023944120045	146	61990415093876
47	5618445228564793	97	66714881583314335	147	51701385089887
48	7140942201229333	98	61033240145235763	148	43082666091665
49	8984922304030443	99	55663091133973346	149	35870481552300

150	29843343433392	200	1526313637	250	37965
151	24808172866872	201	1239515641	251	31193
152	20607116162379	202	1007719386	252	26131
153	17101443169235	203	818568928	253	21406
154	14181008701762	204	666193896	254	17422
155	11747089496422	205	542061609	255	13613
156	9723545122578	206	442072320	256	10696
157	8040378083433	207	360234562	257	7975
158	6644452641044	208	293886739	258	6005
159	5486702080236	209	239045260	259	4374
160	4529003381568	210	194337731	260	3534
161	3736165201688	211	157306293	261	3016
162	3081105018158	212	127325163	262	2635
163	2539255963377	213	102799565	263	2264
164	2091793858275	214	83194097	264	1933
165	1721930513702	215	67300605	265	1603
166	1416734360140	216	54691522	266	1323
167	1164733232308	217	44477808	267	1045
168	957190045595	218	36317458	268	810
169	785911852914	219	29606794	269	585
170	645295369580	220	24117404	270	406
171	529489941608	221	19554213	271	277
172	434606120455	222	15820964	272	258
173	356481490646	223	12736481	273	227
174	292487050484	224	10258846	274	206
175	239755303889	225	8244157	275	173
176	196550315542	226	6659561	276	150
177	160954253448	227	5381526	277	115
178	131791387388	228	4385243	278	90
179	107847709116	229	3576841	279	53
180	88241591630	230	2930385	280	26
181	72162948863	231	2376760	281	15
182	59038079745	232	1924226	282	15
183	48284335855	233	1541327	283	14
184	39509743432	234	1231527	284	14
185	32308399043	235	975760	285	13
186	26423428886	236	777090	286	13
187	21582203262	237	617547	287	12
188	17624621529	238	498228	288	12
189	14368737009	239	404981	289	11
190	11720626558	240	335065	290	11
191	9552812749	241	275998	291	1
192	7790240907	242	226966	292	1
193	6351933169	243	183727	293	1
194	5185250585	244	148442	294	1
195	4232118751	245	117291	295	1
196	3457204258	246	93525	296	1
197	2821392492	247	73010	297	1
198	2302090127	248	57960	298	1
199	1874802017	249	45826	299	1
				300	1

REFERENCE

1. C. N. Cooper and R. E. Kennedy. Is the Mean Bowling Score Awful? *J. Rec. Math.* 18(3) (1985-86).