A KNOCKOUT TOURNAMENT PROBLEM

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A knockout tournament between 2^n players [2, 3] is conducted as follows:

In the first round the 2^n players are split into 2^{n-1} pairs who play each other. The 2^{n-1} winners proceed to the second round and play each other in pairs. The 2^{n-2} winners proceed to the third round, and the process repeats. The one winner of the nth round is declared the winner of the tournament.

A problem in [1] asks the question: What chance has a given player of winning a knockout tournament involving 2^n players?

This paper seeks to answer that question. To that end, we introduce some concepts. A *tournament* T_n is a set A_n of 2^n players, together with the results of the $2^n - 1$ matches between players according to the scheme described above. Let w be the winner of the tournament, w having beaten x in the final round. Then T_n can be thought of as two sub-tournaments, each having 2^{n-1} players, T_n^w won by w and T_n^x won by x, and a final match where w beats x.

Let M be the set of all possible tournaments T_n for a given set A_n of players. Let p_{ij} denote the probability that player a_i beats player a_j in a match. Also, let $p(T_n)$ be the probability that tournament T_n has its stated outcome (including all intermediate games). We see that

$$p(T_n) = \prod_B p_{ij},$$

where $p_{ij} \in B$ if and only if player a_i beats a_j in the tournament T_n.

Another convenient quantity for our discussion is s_n, defined by

$$s_1 = 1; \qquad s_n = \tfrac{1}{2}\binom{2^n}{2^{n-1}}(s_{n-1})^2, \qquad n \geq 2.$$

We then have, by mathematical induction,

$$s_n = (2^n - 1)!\cdot 2^{n+1-2^n}.$$

LEMMA 1. Let T_n be a tournament and T'_n the same tournament except that the result of the final match is reversed. Then

$$p(T_n) + p(T'_n) = p(T_n^w)\cdot p(T_n^x).$$

Proof. This formula follows from the fact that $p_{ij} + p_{ji} = 1$.

THEOREM 1. We have

$$s_n = \sum_{T_n \in M} p(T_n).$$

Proof. We use mathematical induction on n. The case $n = 1$ is obvious. So assume the theorem is true for $n = k-1$, and let

$M' = \{T\colon T$ is a knockout tournament involving 2^{k-1} players from $A_k\}$.

Then we have

$$\sum_{T_k \in M} p(T_k) = \tfrac{1}{2} \sum_{T_k \in M} p(T_k^w)\cdot p(T_k^x), \text{ by Lemma 1}$$

$$= \tfrac{1}{2} s_{k-1} \sum_{T \in M'} p(T), \quad \text{by induction hypothesis}$$

$$= \tfrac{1}{2} s_{k-1} \binom{2^k}{2^{k-1}} s_{k-1}, \text{ by induction hypothesis}$$

$$= s_k.$$

LEMMA 2. The probability that T_n occurs, assuming that the contestants in each tournament are chosen at random, is $p(T_n)/s_n$.

Proof. We again use induction. The case where $n = 1$ is trivial. So assume the formula true for $n = k-1$. Then the probability that T_k occurs is given by

$$\frac{2}{\binom{2^k}{2^{k-1}}} \cdot \frac{p(T_k^w)}{s_{k-1}} \cdot \frac{p(T_k^x)}{s_{k-1}} \cdot p_{wx} = \frac{p(T_k)}{s_k}. \quad \square$$

We are now ready to give the answer to the stated problem, which follows immediately from Lemma 2.

THEOREM 2. The probability that player a_i wins a knockout tournament is given by

$$\sum_{T_n \in M_i} p(T_n)/s_n,$$

where $M_i = \{T_n \in M\colon a_i$ wins $T_n\}$.

As an example of the computations involved in the final formula, we let $p_{ij} = \frac{1}{2}$ for all players in A_n and verify that our formula leads to the expected result, $1/2^n$. The number of elements in M_i is $(2^n - 1)!$ and, for each T_n,

$$p(T_n) = (\tfrac{1}{2})^{2^n - 1}.$$

Thus

$$\sum_{T_n \in M_i} \frac{p(T_n)}{s_n} = \frac{(\frac{1}{2})^{2^n-1}\cdot(2^n-1)!}{(2^n-1)!\cdot 2^{n+1-2^n}} = \frac{1}{2^n}.$$

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REFERENCES

1. H.A. David, *The Method of Paired Comparisons*, London, Griffith, 1963, pp. 82-89.

2. J.A. Hartigan, "Probabilistic Completion of a Knockout Tournament", *Annals of Mathematical Statistics*, 37 (1966) 495-503.

3. J.W. Moon, *Topics on Tournaments*, Holt, Rinehart & Winston, New York, 1968, pp. 47-49.

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POSTSCRIPT TO "AN INTERESTING RECURSIVE FUNCTION"

I found out, just too late for inclusion in my note published here last month [1982: 69], that the conjecture "the sequence will eventually enter a cycle containing 000...0*XX*" is invalid. Accordingly, the last paragraph of the note should be amended to read as follows:

Readers are invited to show that, if n_0 has any number of digits, the sequence will eventually enter a cycle. Can you describe a general component of this cycle? It is conjectured (but this is only a wild conjecture with little evidence to support it) that one can guarantee that $n_0 \to \ldots \to 000\ldots00$ for all k-digit n_0 only if k is a power of 2 .

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THE PUZZLE CORNER

Puzzle No. 15: Heteronyms (2, 5; 3, 4; 7)

Professor Matrix is MY LAST
For mental mathematics.
Extracting roots? He's very fast.
At sight, he solves quadratics,
Does logarithms with MY FIRST,
And "crypts" as if they're all plain text;
But here is where he's not well versed —
To check his check book, he's MY NEXT.

ALAN WAYNE, Holiday, Florida