44	19	58	9	22	7
47	56	21	6	59	10
20	41	12	57	8	23
53	48	5	24	11	60
32	13	40	61	34	25
49	4	33	28	37	62
14	29	64	39	26	35
1	50	27	36	63	38

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IS THE MEAN BOWLING SCORE AWFUL?

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As the title of this article suggests, in the following discussion we are concerned with the game of bowling. We will be primarily interested in the process of keeping score. We will not be able to help you improve your bowling average, but we might make you feel better about your average by calculating the mean bowling score of all possible games.

To clarify this, we will demonstrate how to calculate the score for a particular bowling game. Consider the sequence of frames:

FRAME	1	2	3	4	5	6	7	8	9	10
	3 4	6 /	4 0	X	3 /	X	2 5	X	X	3 / 4
	7	21	25	45	65	82	89	112	132	146

(1)

The lower number in each frame is the cumulative score. The upper numbers in each frame are the number of pins knocked down by the first and second ball, respectively. The "/" indicates that the remaining pins were knocked down by the second ball and is called a "spare." The "X" indicates that all ten pins were knocked down by the first ball and is called a "strike." The rules for calculating the cumulative score for a given frame are: (2)

- For frames one through nine:
  - (i) If neither a spare nor a strike is thrown, then the total number of pins knocked down by the two balls is added to the previous cumulative score.

(ii) If a spare is thrown, then the number of pins knocked down by the next ball thrown, plus 10, is added to the previous cumulative score.
(iii) If a strike is thrown, then the number of pins knocked down by the next two balls thrown, plus 10, is added to the previous cumulative score.

- For the tenth frame:
  (iv) The total number of pins knocked down is added to the previous cumulative score. This is enhanced by the convention that if a spare is thrown in the tenth frame, an extra ball is allowed, while if a strike is thrown, two extra balls are given as a bonus.

Most of these possibilities are exemplified in (1). Frame seven demonstrates (i), frame two demonstrates (ii), frame six demonstrates (iii), while frame ten demonstrates (iv) and the bonus given by making a spare in the tenth frame.

The final score of 146 is probably not a "bad" score for a person who bowls only occasionally, but is far from the "perfect" score of 300. Note that the game given by (1) can be represented by the sequence of nine ordered pairs and the ordered triple,

$$(3,4), (6,4), (4,0), (10,0), (3,7), (10,0), (2,5), (10,0), (10,0), (3,7,4). \tag{3}$$

With the above example as a model, we define a "bowling game," g, as a sequence of the form

$$(x_1,y_1), (x_2,y_2), \ldots, (x_9,y_9), (x_{10},y_{10},z_{10}), \tag{4}$$

where the terms of (4) represent the ten frames of a game and each component of a term of (4) denotes the number of pins knocked down by that ball. In what follows we use the following sets to facilitate the determination of the arithmetic mean of the scores of all possible bowling games. Here, all variables are non-negative integers.

$$A = \left\{ (x,y) : x + y \leqslant 10 \right\}, \tag{5}$$

$$\begin{aligned} B = {} & \left\{ (x,y,0) : x + y \leqslant 9 \right\} \\ & \cup \left\{ (x,10-x,z) : x \leqslant 9 \text{ and } z \leqslant 10 \right\} \\ & \cup \left\{ (10,y,z) : y \leqslant 9 \text{ and } y + z \leqslant 10 \right\} \\ & \cup \left\{ (10,10,z) : z \leqslant 10 \right\}, \end{aligned}$$

and

G = the set of all possible bowling games.

Note that the first nine frames of any game are elements of A while the tenth frame is an element of B. Since the number of elements in A is 66 and the number of elements in B is 241, it follows that the number of elements in G is

$$\#(G) = (66^9)(241), \tag{6}$$

which is approximately 5.7 billion billion.

For $i = 1, 2, 3, 4, 5, 6, 7$, and 8, consider the expression

$$x_i + y_i + \left[\frac{x_i + y_i}{10}\right] x_{i+1} + \left[\frac{x_i}{10}\right] y_{i+1} + \left[\frac{x_i + x_{i+1}}{20}\right] x_{i+2} \qquad (7)$$

As usual, the square brackets indicate the integral part operator. We see that (7) is the number of pins added to the previous cumulative score to arrive at the cumulative score in frame i since (7) takes into consideration the definitions of sets A and B and the scoring rules (i), (ii), and (iii) given in (2). Likewise, the expression

$$x_9 + y_9 + \left[\frac{x_9 + y_9}{10}\right] x_{10} + \left[\frac{x_9}{10}\right] y_{10} + x_{10} + y_{10} + z_{10} \qquad (8)$$

is the number of pins added to the cumulative score of frame eight to give the final score, since (8) takes into consideration the definitions of A and B and the scoring rules (i), (ii), (iii), and (iv) given in (2).

Thus, the score, $s(g)$, of the game given by (4) is determined by the formula

$$\sum_{i=1}^{\infty} (x_i + y_i + \left[\frac{x_i + y_i}{10}\right] x_{i+1} + \left[\frac{x_i}{10}\right] y_{i+1} + \left[\frac{x_i + x_{i+1}}{20}\right] x_{i+2}$$
$$+ x_9 + y_9 + \left[\frac{x_9 + y_9}{10}\right] x_{10} + \left[\frac{x_9}{10}\right] y_{10} + x_{10} + y_{10} + z_{10} \,. \qquad (9)$$

We now proceed to calculate the arithmetic mean of the scores of all possible games. That is, we will determine the value of the quotient,

$$\frac{\sum_{g \in G} s(g)}{\#(G)} \,. \qquad (10)$$

To compute the numerator of (10), we will need the following equalities which follow immediately from the definitions of sets A and B. Hence, their derivation will not be given here.

$$\sum_{(x,y) \in A} x = \sum_{(x,y) \in A} y = 220, \qquad \sum_{(x,y,z) \in B} x = 1420, \qquad (11)$$

$$\sum_{(x,y,z) \in B} y = 1090, \qquad \text{and} \qquad \sum_{(x,y,z) \in B} z = 825 \,.$$

Then by formula (9), after regrouping and changing the order of summation, we have

$$\sum_{g \in G} s(g) = \sum_{i=1}^{9} \sum_{g \in G} (x_i + y_i) + \sum_{i=1}^{8} \sum_{g \in G} \left[\frac{x_i + y_i}{10}\right] x_{i+1} \qquad (12)$$
$$+ \sum_{i=1}^{8} \sum_{g \in G} \cdot \frac{x_i}{10} \cdot y_{i+1} + \sum_{i=1}^{8} \sum_{g \in G} \left[\frac{x_i + x_{i+1}}{20}\right] x_{i+2}$$

$$+ \sum_{g\in G}\left[\frac{x_9+y_9}{10}\right]x_{10} + \sum_{g\in G}\left[\frac{x_9}{10}\right]y_{10} + \sum_{g\in G}(x_{10}+y_{10}+z_{10}).$$

From (11), and the definitions of sets A, B, and G, it follows that for $i = 1, 2, 3, \ldots, 9$,

$$\sum_{g\in G}(x_i+y_i) = 66^8(241)(440), \tag{13}$$

and for $i = 1, 2, 3, \ldots, 8$,

$$\sum_{g\in G}\left[\frac{x_i+y_i}{10}\right]x_{i+1} = 66^7(241)(11)(220),$$

and (14)

$$\sum_{g\in G}\left[\frac{x_i}{10}\right]y_{i+1} = 66^7(241)(220),$$

and for $i = 1, 2, 3, \ldots, 7$,

$$\sum_{g\in G}\left[\frac{x_i+x_{i+1}}{20}\right]x_{i+1} = 66^6(241)(220). \tag{15}$$

The remaining terms of (12), which represent the tenth frame, are given by (16)

$$\sum_{g\in G}\left[\frac{x_8+x_9}{20}\right]x_{10} = 66^7(1420), \qquad \sum_{g\in G}\left[\frac{x_9+y_9}{10}\right]x_{10} = 66^8(11)(1420),$$

$$\sum_{g\in G}\left[\frac{x_9}{10}\right]y_{10} = 66^8(1090), \qquad \text{and} \quad \sum_{g\in G}(x_{10}+y_{10}+z_{10}) = 66^9(3335).$$

Thus, from (13), (14), (15), and (16), we have that

$$\frac{\sum_{g\in G} s(g)}{\#(G)} = \frac{\begin{array}{c}9(66^8)(241)(440) + (8)(66^7)(241)(11)(220)\\ + 8(66^7)(241)(220) + (7)(66^6)(241)(220)\\ + 66^7(1420) + 66^8(11)(1420) + 66^8(1090) + 66^9(3335)\end{array}}{66^9(241)}, \tag{17}$$

which is approximately 80 (for precision bowlers, it's 79.7439 . . .).

Thus, the mean bowling score is indeed awful even if you are just an occasional bowler. Even though this information is interesting, there are more difficult questions about the game of bowling that could be asked. For example, you might wish to determine the standard deviation of the set of bowling scores and hence know more about the distribution of the set of all bowling scores. But the exact determination of the distribution of the set of scores is, in our opinion, a difficult problem. For example, given an integer k between 0 and 300, how many different bowling games have the score k? This, we leave as an open problem.

NOTEWORTHY NUMBE

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The number 2469 is particularl
of the consecutive multiples, 4
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repdigit.

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However, there are other int
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as exponents while retaining th
number. Thus,

$$\begin{aligned} 127 &= -1 + 2^7 \\ 145 &= 1! + 4! + 5! \\ 343 &= (3+4)^3 \\ 3125 &= (3^1+2)^5 \\ 4374 &= 4 \times 3^7 \div \sqrt{4} \end{aligned}$$

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