THE FIRST MOMENT OF THE NUMBER OF 1'S FUNCTION IN THE β-EXPANSION OF THE POSITIVE INTEGERS

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Abstract. The first moment of the number of 1's function in the $\beta = (1 + \sqrt{5})/2$-expansion of the positive integers will be studied.

1. Introduction. In [2], Coquet and van den Bosch studied the first moment of the number of 1's function in the Zeckendorf decomposition [3,5] of the positive integers.

The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. The Zeckendorf decomposition of a natural number n is the unique expression of n as a sum of Fibonacci numbers with nonconsecutive indices and with each index greater than 1. For example, the Zeckendorf decomposition of one million is

$$1000000 = 832040 + 121393 + 46368 + 144 + 55$$

$$= F_{30} + F_{26} + F_{24} + F_{12} + F_{10}.$$

Therefore, if we have

$$n = \sum_{k \geq 2}^{m} b_k F_k$$

then we write $n = b_m b_{m-1} \ldots b_{2_Z}$. Thus,

$$1000000 = 10001010000000000010100000000_Z.$$

Representing numbers by their Zeckendorf decomposition is referred to as the Zeckendorf number system. The representation of the first 56 natural numbers in the Zeckendorf number system is given in the following table.

$1 =$	1_Z	$29 =$	1010000_Z
$2 =$	10_Z	$30 =$	1010001_Z
$3 =$	100_Z	$31 =$	1010010_Z
$4 =$	101_Z	$32 =$	1010100_Z
$5 =$	1000_Z	$33 =$	1010101_Z
$6 =$	1001_Z	$34 =$	10000000_Z
$7 =$	1010_Z	$35 =$	10000001_Z
$8 =$	10000_Z	$36 =$	10000010_Z
$9 =$	10001_Z	$37 =$	10000100_Z
$10 =$	10010_Z	$38 =$	10000101_Z
$11 =$	10100_Z	$39 =$	10001000_Z
$12 =$	10101_Z	$40 =$	10001001_Z
$13 =$	100000_Z	$41 =$	10001010_Z
$14 =$	100001_Z	$42 =$	10010000_Z
$15 =$	100010_Z	$43 =$	10010001_Z
$16 =$	100100_Z	$44 =$	10010010_Z
$17 =$	100101_Z	$45 =$	10010100_Z
$18 =$	101000_Z	$46 =$	10010101_Z
$19 =$	101001_Z	$47 =$	10100000_Z
$20 =$	101010_Z	$48 =$	10100001_Z
$21 =$	1000000_Z	$49 =$	10100010_Z
$22 =$	1000001_Z	$50 =$	10100100_Z
$23 =$	1000010_Z	$51 =$	10100101_Z
$24 =$	1000100_Z	$52 =$	10101000_Z
$25 =$	1000101_Z	$53 =$	10101001_Z
$26 =$	1001000_Z	$54 =$	10101010_Z
$27 =$	1001001_Z	$55 =$	100000000_Z
$28 =$	1001010_Z	$56 =$	100000001_Z

The Zeckendorf Decomposition of the First 56 Natural Numbers.

Definition 1. Let n be a positive integer. Then $s_Z(n)$ is defined to be the digital sum of (number of 1's in) the Zeckendorf decomposition of the natural number n.

For example, $s_Z(56) = 2$ and $s_Z(1000000) = 5$.

Coquet and van den Bosch [2] found that for any positive integer n,

$$\sum_{k<F_n} s_Z(k) = \frac{5-9\sqrt{5}}{50}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n (n+1)$$
$$+ \frac{5+9\sqrt{5}}{50}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{-1-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n (n+1), \qquad (1)$$

and also for any positive x,

$$\sum_{k<x} s_Z(k) = \frac{3-\beta}{5\log\beta} x\log x + xG\left(\frac{\log x}{\log\beta}\right) + O(\log x). \qquad (2)$$

Here, $\beta = (1+\sqrt{5})/2$, $\log x$ denotes the natural logarithm of x, and $G\colon \mathbb{R} \to \mathbb{R}$ is a continuous, nowhere differentiable function of period 1. In this paper we will study a sum similiar to (1) involving the β-expansion [1,4] of the positive integers.

2. β-Expansion. The Lucas sequence is defined as follows. Let $L_0 = 2$, $L_1 = 1$, and $L_i = L_{i-1} + L_{i-2}$ for $i \geq 2$. Let $\beta = (1+\sqrt{5})/2$. The β–expansion of a natural number n is the unique finite sum of integral, nonconsecutive powers of β that equals n. For example, the β-expansion of one million is

$$\begin{aligned}
1000000 &= \frac{710647-317811\sqrt{5}}{2} + \frac{271443-121393\sqrt{5}}{2} + \frac{15127-6765\sqrt{5}}{2} \\
&+ \frac{2207-987\sqrt{5}}{2} + \frac{843-377\sqrt{5}}{2} + \frac{-199+89\sqrt{5}}{2} \\
&+ \frac{-76+34\sqrt{5}}{2} + \frac{7-3\sqrt{5}}{2} + \frac{2+0\sqrt{5}}{2} + \frac{7+3\sqrt{5}}{2} \\
&+ \frac{47+21\sqrt{5}}{2} + \frac{521+233\sqrt{5}}{2} + \frac{2207+987\sqrt{5}}{2} \\
&+ \frac{15127+6765\sqrt{5}}{2} + \frac{271443+121393\sqrt{5}}{2} + \frac{710647+317811\sqrt{5}}{2} \\
&= \beta^{-28} + \beta^{-26} + \beta^{-20} + \beta^{-16} + \beta^{-14} \\
&+ \beta^{-11} + \beta^{-9} + \beta^{-4} + \beta^{0} + \beta^{4} + \beta^{8} \\
&+ \beta^{13} + \beta^{16} + \beta^{20} + \beta^{26} + \beta^{28}.
\end{aligned}$$

Therefore, if we have

$$n = \sum_{k=-m}^{m} b_k \beta^k$$

then we write $n = b_{-m} b_{-m+1} \ldots b_{-1} \underline{b_0} b_1 \ldots b_{m-1} b_{m\beta}$. Note that the coefficient of β^0 is underlined. Thus,

$$1000000 = 1010000010001010010100001000\underline{1}0001000100001001000100000101_{\beta}$$

is the β-expansion of 1000000. Representing numbers in their β-expansion will be referred to as the base β number system. The representation of the first 56 natural numbers in base β is given in the table below.

1 =									$\underline{1}$								
2 =							1	0	$\underline{0}$	1							
3 =							1	0	$\underline{0}$	0	1						
4 =							1	0	$\underline{1}$	0	1						
5 =					1	0	0	1	$\underline{0}$	0	0	1					
6 =					1	0	0	0	$\underline{0}$	1	0	1					
7 =					1	0	0	0	$\underline{0}$	0	0	0	1				
8 =					1	0	0	0	$\underline{1}$	0	0	0	1				
9 =					1	0	1	0	$\underline{0}$	1	0	0	1				
10 =					1	0	1	0	$\underline{0}$	0	1	0	1				
11 =					1	0	1	0	$\underline{1}$	0	1	0	1				
12 =			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1			
13 =			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1			
14 =			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1			
15 =			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1			
16 =			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1			
17 =			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1			
18 =			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1		
19 =			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1		
20 =			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1		
21 =			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1		
22 =			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1		
23 =			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1		
24 =			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1		
25 =			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1		
26 =			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1		
27 =			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1		
28 =			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1		
29 =			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1		
30 =	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1	
31 =	1	0	0	1	0	1	0	0	$\underline{0}$	1	0	0	0	0	0	1	
32 =	1	0	0	1	0	1	0	0	$\underline{0}$	0	1	0	0	0	0	1	
33 =	1	0	0	1	0	1	0	0	$\underline{1}$	0	1	0	0	0	0	1	
34 =	1	0	0	1	0	0	0	1	$\underline{0}$	0	0	1	0	0	0	1	
35 =	1	0	0	1	0	0	0	0	$\underline{0}$	1	0	1	0	0	0	1	
36 =	1	0	0	1	0	0	0	0	$\underline{0}$	0	0	0	1	0	0	1	
37 =	1	0	0	1	0	0	0	0	$\underline{1}$	0	0	0	1	0	0	1	
38 =	1	0	0	1	0	0	1	0	$\underline{0}$	1	0	0	1	0	0	1	
39 =	1	0	0	1	0	0	1	0	$\underline{0}$	0	1	0	1	0	0	1	
40 =	1	0	0	1	0	0	1	0	$\underline{1}$	0	1	0	1	0	0	1	
41 =	1	0	0	0	0	1	0	1	$\underline{0}$	0	0	0	0	1	0	1	
42 =	1	0	0	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	0	1	
43 =	1	0	0	0	0	1	0	0	$\underline{0}$	0	1	0	0	1	0	1	
44 =	1	0	0	0	0	1	0	0	$\underline{1}$	0	1	0	0	1	0	1	
45 =	1	0	0	0	0	0	0	1	$\underline{0}$	0	0	1	0	1	0	1	
46 =	1	0	0	0	0	0	0	0	$\underline{0}$	1	0	1	0	1	0	1	
47 =	1	0	0	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	0	0	1
48 =	1	0	0	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	0	0	1
49 =	1	0	0	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	0	0	1
50 =	1	0	0	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	0	0	1
51 =	1	0	0	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	0	0	1
52 =	1	0	0	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	0	0	1
53 =	1	0	0	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	0	0	1
54 =	1	0	0	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	0	0	1
55 =	1	0	0	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	0	0	1
56 =	1	0	0	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	0	0	1

The Base β Representation of the First 56 Natural Numbers.

Definition 2. Let n be a positive integer. Then $s_\beta(n)$ is defined to be the number of 1's in the β-expansion of the natural number n.

For example, $s_\beta(56) = 6$ and $s_\beta(1000000) = 16$.

Definition 3. Let

$$\beta_n = \sum_{k \le L_n} s_\beta(k).$$

3. Results.

Lemma 1. Let n be an integer. Then

$$\beta^n = \frac{L_n + F_n\sqrt{5}}{2}.$$

Here, for $n > 0$,

$$F_{-n} = (-1)^{n+1} F_n \text{ and } L_{-n} = (-1)^n L_n.$$

Lemma 2.

$$L_{2n} = 1\underbrace{0\cdots0}_{2n-1}\underline{0}\underbrace{0\cdots0}_{2n-1}1_\beta \text{ for } n \ge 1,$$

$$L_{2n-1} = \underbrace{10\cdots10}_{n-1}\underline{1}\underbrace{01\cdots01}_{n-1}{}_\beta \text{ for } n \ge 2,$$

$$L_{2n-1} + 1 = 10\underbrace{01\cdots01}_{n-1}\underline{0}\underbrace{0\cdots0}_{2n-2}1_\beta \text{ for } n \ge 2,$$

$$2L_{2n-2} = 1001\underbrace{0\cdots0}_{2n-4}\underline{0}\underbrace{0\cdots0}_{2n-5}1001_\beta \text{ for } n \ge 3,$$

$$L_{2n-1} + L_{2n-3} = 100100\underbrace{10\cdots10}_{n-3}\underline{1}\underbrace{01\cdots01}_{n-2}001_\beta \text{ for } n \ge 3,$$

$$L_{2n-1} + L_{2n-3} + 1 = 1000\underbrace{01\cdots01}_{n-2}\underline{0}\underbrace{0\cdots0}_{2n-4}101_\beta \text{ for } n \ge 3.$$

Lemma 3. $\beta_1 = 1$, $\beta_2 = 5$, $\beta_3 = 8$, $\beta_4 = 16$, and $\beta_5 = 32$. For $n \geq 3$,

$$\beta_{2n} = \beta_{2n-1} + 2\beta_{2n-2} - 2\beta_{2n-3} + \beta_{2n-5} + 2L_{2n-1} - 4L_{2n-4}$$

and

$$\beta_{2n+1} = \beta_{2n} + \beta_{2n-1} + 2L_{2n-1}.$$

Theorem. Let n be a positive integer. Then

$$\begin{aligned}\beta_n &= \sum_{k \leq L_n} s_\beta(k) \\ &= \frac{3}{2} - \frac{3}{2}(-1)^n + \frac{1-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &+ \frac{5-\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n (n+1) + \frac{5+\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n (n+1)\end{aligned}$$

4. Proofs.

Proof of Lemma 1. By induction on n (both ways).

Proof of Lemma 2. Lemma 2 is Lemma 3.2 and 3.3 in [4].

Proof of Lemma 3. We will first prove that

$$\beta_{2n+1} = \beta_{2n} + \beta_{2n-1} + 2L_{2n-1}.$$

We first note that

$$\beta_{2n+1} = \beta_{2n} + \sum_{k=L_{2n}+1}^{L_{2n+1}} s_\beta(k).$$

Since by Lemma 2

$$L_{2n} = 1\underbrace{0\cdots0}_{2n-1}\underline{0}\underbrace{0\cdots0}_{2n-1}1_\beta,$$

$$L_{2n} = \beta^{-2n} + \beta^{2n}.$$

Hence for $1 \leq k \leq L_{2n-1}$, the base β representation of $L_{2n} + k$ is the base β representation of k plus β^{-2n} plus β^{2n}. The recurrence relation follows.

Next, we will prove that

$$\beta_{2n} = \beta_{2n-1} + 2\beta_{2n-2} - 2\beta_{2n-3} + \beta_{2n-5} + 2L_{2n-1} - 4L_{2n-4}.$$

We note that

$$\beta_{2n} = \beta_{2n-1} + \sum_{k=L_{2n-1}+1}^{L_{2n}} s_\beta(k).$$

We first note that the base β representation of all the numbers from $L_{2n-1}+1$ to $L_{2n}-1$ include the expression

$$\beta^{-2n} + \beta^{2n-1}$$

and the base β representation of L_{2n} includes the expression

$$\beta^{-2n} + \beta^{2n}.$$

There are L_{2n-2} numbers in this list. In addition, the base β representation of $L_{2n-1}+1$ to $L_{2n-1}+L_{2n-3}$ includes the expression

$$\beta^{-2n+3}.$$

Also, the base β representation of $2L_{2n-2}$ to $L_{2n-1}+L_{2n-3}$ and $L_{2n-1}+L_{2n-3}+1$ to $L_{2n}-1$ includes the expression

$$\beta^{2n-4} \text{ and } \beta^{2n-3}, \text{ respectively.}$$

In each of these lists there are L_{2n-3} numbers. Note that the base β representation of L_{2n} is precisely

$$\beta^{-2n} + \beta^{2n}.$$

Counting the number of terms above gives

$$2L_{2n-2} + 2L_{2n-3} = 2L_{2n-1}.$$

Next, we wish to examine the middle terms of the base β representations of $L_{2n-1}+1$ to L_{2n}. We will divide these numbers into three parts; $L_{2n-1}+1$ to $2L_{2n-2}$, $2L_{2n-2}+1$ to $L_{2n-1}+L_{2n-3}$, and $L_{2n-1}+L_{2n-3}+1$ to L_{2n}.

Consider the first group $L_{2n-1}+1$ to $2L_{2n-2}$. There are L_{2n-4} numbers in this group. Let $1 \le k < L_{2n-4}$. To investigate the middle terms in the base β representation of $L_{2n-1}+k$, we will compare these terms to the middle terms in the base β representation of $L_{2n-3}+k$. (There are no 1's in the middle terms in $L_{2n-1}+L_{2n-4}$ and $L_{2n-3}+L_{2n-4} = L_{2n-2}$ so we don't need to worry about $k = L_{2n-4}$.) By Lemma 2,

$$L_{2n-1} + k = \beta^{-2n} + \beta^{-2n+3} + x + \beta^{2n-1}$$

$$L_{2n-3} + k = \beta^{-2n+2} + y + \beta^{2n-3}$$

where x and y are unknown middle expressions involving powers of β. However, by comparing the expressions

$$x = L_{2n-1} + k - \beta^{-2n} - \beta^{-2n+3} - \beta^{2n-1}$$

$$y = L_{2n-3} + k - \beta^{2n-3} - \beta^{-2n+2}$$

and using Lemma 1, it follows that the two right-hand sides of the above equations are equal. Therefore, the base β representations of x and y are equal. Since we know the number of 1's in the base β representation of y, we know the number of 1's in the base β representation of x. Therefore, the number of 1's in the middle terms of this first group is

$$\beta_{2n-2} - \beta_{2n-3} - 2L_{2n-4}.$$

The third group is analogous to the first. The third group consists of the numbers $L_{2n-3} + L_{2n-1} + 1$ to L_{2n}. There are L_{2n-4} numbers in this group. Let $1 \leq k < L_{2n-4}$. To investigate the middle terms in the base β representation of $L_{2n-3} + L_{2n-1} + k$, we will compare these terms to the middle terms in the base β representation of $L_{2n-3} + k$. (Again, there are no 1's in the middle terms in $L_{2n-1} + L_{2n-4}$ and L_{2n} so we don't need to worry about $k = L_{2n-4}$.) By Lemma 2,

$$L_{2n-3} + L_{2n-1} + k = \beta^{-2n} + x + \beta^{2n-3} + \beta^{2n-1}$$

$$L_{2n-3} + k = \beta^{-2n+2} + y + \beta^{2n-3}$$

where x and y are unknown middle expressions involving powers of β. However, by comparing the expressions

$$x = L_{2n-3} + L_{2n-1} + k - \beta^{-2n} - \beta^{2n-3} - \beta^{2n-1}$$

$$y = L_{2n-3} + k - \beta^{-2n+2} - \beta^{2n-3}$$

and using Lemma 1, it follows that the two right-hand sides of the above equations are equal. Therefore, the base β representations of x and y are equal. Since we know the number of 1's in the base β representation of y, we know the number of

1's in the base β representation of x. Therefore, the number of 1's in the middle terms of this first group is

$$\beta_{2n-2} - \beta_{2n-3} - 2L_{2n-4}.$$

The second group is the last one we have to do. This group consists of the middle terms of $2L_{2n-2} + 1$ to $L_{2n-1} + L_{2n-3}$. There are L_{2n-5} numbers in this group. Since by Lemma 2 we have

$$2L_{2n-2} = 1001\underbrace{0\cdots0}_{2n-4}\underline{0}\underbrace{0\cdots0}_{2n-5}1001_\beta,$$

it follows

$$2L_{2n-2} = \beta^{-2n} + \beta^{-2n+3} + \beta^{2n-4} + \beta^{2n-1}.$$

Now let $1 \leq k \leq L_{2n-5}$. To investigate the middle terms in the base β representation of $2L_{2n-2}+k$, we will compare these terms to the base β representation of k. Because of absence of β terms in the middle of $2L_{2n-2}$, the middle terms in $L_{2n-2} + k$ are precisely the base β representation of k. Therefore the number of 1's in the middle terms of the second group is

$$\beta_{2n-5}.$$

Putting this all together we have that

$$\begin{aligned}\beta_{2n} = \beta_{2n-1} \\ &+ 2L_{2n-1} \\ &+ 2(\beta_{2n-2} - \beta_{2n-3} - 2L_{2n-4}) \\ &+ \beta_{2n-5}.\end{aligned}$$

The result follows.

Proof of the Theorem. From Lemma 3 we have that $\beta_1 = 1$, $\beta_2 = 5$, $\beta_3 = 8$, $\beta_4 = 16$, $\beta_5 = 32$ and for $n \geq 3$,

$$\beta_{2n} = \beta_{2n-1} + 2\beta_{2n-2} - 2\beta_{2n-3} + \beta_{2n-5} + 2L_{2n-1} - 4L_{2n-4}$$

and

$$\beta_{2n+1} = \beta_{2n} + \beta_{2n-1} + 2L_{2n-1}.$$

Let

$$\beta(x) = \sum_{n \geq 1} \beta_n x^n$$

and

$$L(x) = \sum_{n \geq 0} L_n x^n.$$

We note that

$$\sum_{n \geq 1} \beta_{2n} x^{2n} = \frac{\beta(x) + \beta(-x)}{2}$$

and

$$\sum_{n \geq 1} \beta_{2n-1} x^{2n-1} = \frac{\beta(x) - \beta(-x)}{2}.$$

Therefore, it follows that the odd recurrence relation yields

$$\left(\frac{1}{2} - \frac{x}{2} - \frac{x^2}{2}\right)\beta(x) + \left(-\frac{1}{2} - \frac{x}{2} + \frac{x^2}{2}\right)\beta(-x) = x + 2x^2\left(\frac{L(x) - L(-x)}{2}\right).$$

The even recurrence relation produces

$$\begin{aligned} &\left(\frac{1}{2} - \frac{x}{2} - x^2 + x^3 - \frac{x^5}{2}\right)\beta(x) \\ &+ \left(\frac{1}{2} + \frac{x}{2} - x^2 - x^3 + \frac{x^5}{2}\right)\beta(-x) \\ &= 2x^2 + 2x\left(\frac{L(x) - L(-x)}{2}\right) - 4x^4\left(\frac{L(x) + L(-x)}{2}\right). \end{aligned}$$

Using the fact that

$$L(x) = \frac{2-x}{1-x-x^2}$$

we can solve these two equations in two variables $\beta(x)$ and $\beta(-x)$ simultaneously to obtain

$$\beta(x) = \frac{\frac{x}{2} + \frac{5x^2}{2} + \frac{x^3}{2} - \frac{19x^4}{2} - \frac{7x^5}{2} + 16x^6 + 4x^7 - \frac{21x^8}{2} + \frac{3x^{10}}{2}}{\left(\frac{1}{2} - 2x^2 + 2x^4 - \frac{x^6}{2}\right)(1-x-x^2)(1+x-x^2)}.$$

Simplifying $\beta(x)$ results in

$$\beta(x) = -\frac{x(3x^5+6x^4-6x^3-4x^2+3x+1)}{(-1+x+x^2)^2(x-1)(x+1)}.$$

Expanding this into partial fractions gives

$$\begin{aligned}\beta(x) = -3 - \frac{3}{2(x-1)} - \frac{3}{2(x+1)} + \frac{3-\sqrt{5}}{(2x+1-\sqrt{5})} + \frac{3+\sqrt{5}}{(2x+1+\sqrt{5})} \\ + \frac{40-16\sqrt{5}}{5(2x+1-\sqrt{5})^2} + \frac{40+16\sqrt{5}}{5(2x+1+\sqrt{5})^2}.\end{aligned}$$

Using the power series expansions of the 6 series

$$\begin{aligned}
\frac{1}{2x+1-\sqrt{5}} &= \frac{1}{1-\sqrt{5}} \sum_{n\geq 0} \left(\frac{1+\sqrt{5}}{2}\right)^n x^n, \\
\frac{1}{2x+1+\sqrt{5}} &= \frac{1}{1+\sqrt{5}} \sum_{n\geq 0} \left(\frac{1-\sqrt{5}}{2}\right)^n x^n, \\
\frac{1}{(2x+1-\sqrt{5})^2} &= \frac{1}{(1-\sqrt{5})^2} \sum_{n\geq 0} \left(\frac{1+\sqrt{5}}{2}\right)^n (n+1)x^n, \\
\frac{1}{(2x+1+\sqrt{5})^2} &= \frac{1}{(1+\sqrt{5})^2} \sum_{n\geq 0} \left(\frac{1-\sqrt{5}}{2}\right)^n (n+1)x^n, \\
\frac{1}{x-1} &= \sum_{n\geq 0} -1x^n \text{ and} \\
\frac{1}{x+1} &= \sum_{n\geq 0} (-1)^n x^n,
\end{aligned}$$

we conclude that

$$\beta_n = \frac{3}{2} - \frac{3}{2}(-1)^n + \frac{1-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n$$
$$+ \frac{5-\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n (n+1) + \frac{5+\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n (n+1)$$

In fact, a new recurrence relation which combines both the even and odd parts of the above recurrence relation is $\beta_1 = 1$, $\beta_2 = 5$, $\beta_3 = 8$, $\beta_4 = 16$, $\beta_5 = 32$, $\beta_6 = 59$, and for $n \geq 7$

$$\beta_n = 2\beta_{n-1} + 2\beta_{n-2} - 4\beta_{n-3} - 2\beta_{n-4} + 2\beta_{n-5} + \beta_{n-6}.$$

5. Questions. Several questions remain unanswered. For example, can a closed form formula be discovered for the second moment, i.e.

$$\sum_{k<F_n} s_Z(k)^2 \text{ or } \sum_{k\leq L_n} s_\beta(k)^2?$$

What can be said about higher moments of either the function s_Z or s_β? Finally, can a formula similar to (2) be found for

$$\sum_{k\leq x} s_\beta(k).$$

References

1. G. Bergman, "A Number System With an Irrational Base," *Mathematics Magazine,* 31 (1957), 98–110.
2. J. Coquet and P. van den Bosch, "A Summation Formula Involving Fibonacci Digits," *Journal of Number Theory,* 22 (1986), 139–146.
3. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics,* 2nd ed., Addison-Wesley Publishing Company, Reading, Massachusetts, 1994.
4. E. Hart and L. Sanchis, "On the Occurrence of F_n in the Zeckendorf Decomposition of nF_n, *The Fibonacci Quarterly,* 37.1 (1999), 21–33.
5. C. G. Lekkerkerker, "Voorstelling van Natuurlijke Getallen door een Som van Getallen van Fibonacci," *Simon Stevin,* 29 (1952), 190–195.

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