n-Card Tricks

Hang Chen and Curtis Cooper

Hang Chen (hchen@ucmo.edu) is a professor at the University of Central Missouri. He received his undergraduate degree in China and his Ph.D. from Western Michigan University. His interests include mathematics, computing, bridge, photography, traveling, and tennis.

Curtis Cooper (cooper@ucmo.edu) earned his B.A. from Culver-Stockton College and his M.S. and Ph.D. from Iowa State University. He is a professor at the University of Central Missouri. Curtis enjoys running and tennis and is an avid Nebraska football fan. He is the editor of *The Fibonacci Quarterly* and the problem editor of *The College Mathematics Journal*. He and colleague Steven Boone lead the University's GIMPS team that found the 43rd and 44th Mersenne primes within 9 months,

$$2^{30,402,457} - 1 \quad \text{and} \quad 2^{32,582,657} - 1.$$

In Fitch Cheney's 5-card trick, a volunteer chooses 5 cards at random from a standard deck and hands them to the magician's assistant. The assistant hides one of the cards and shows the other four to the magician, who promptly identifies the hidden card. Properly done, this trick puzzles the viewer who typically cannot conceive how information on the hidden card can possibly have been transmitted to the magician.

Here we extend Cheney's 5-card trick to what we call an n-card trick. An n-card trick works as follows: n cards are chosen by the audience from a standard deck of 52 cards and one card is hidden. The assistant *arranges the cards in a row* (including the hidden card, face-down) and the magician names the hidden card after viewing the arrangement of cards. Note that the *lower* the number n, the more dramatic the trick, because the fewer the number of cards, the less information can be conveyed to the magician.

We shall vary two significant factors: whether the hidden card is chosen by the audience or by the assistant, and whether the magician watches the assistant arrange the cards (timing) or not. We shall determine the minimum number n so that such an n-card trick is possible, and also present several relatively easy n-card tricks for readers to perform. Our purpose is to improve, if that be possible, what some have called the "best card trick" [**2, 3**].

The audience picks the hidden card

Let the audience pick n cards, and then select a single card to hide by turning it face-down, after showing it to the assistant. The assistant arranges the n cards in a row, behind the magician's back. We will consider arrangements with timing later. The magician views the arrangement of cards and names the hidden card.

Here is an example of a 5-card trick. The audience picks five cards and selects the hidden card. The assistant arranges the five cards, as shown, for example, in Figure 1.

Figure 1. The hidden card is the queen of clubs.

The magician and assistant use a pre-agreed upon ordering of the deck of cards to interpret the arrangement. Many orderings are in use today, but suppose the following system is used. The order of cards is given first by an ordering of the ranks: $2 < 3 < \cdots < K < A$; and then, within rank, by an ordering of the suits: $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$. Therefore, the whole deck of cards, in order, is:

$$\clubsuit 2 < \diamondsuit 2 < \heartsuit 2 < \spadesuit 2 < \clubsuit 3 < \diamondsuit 3 < \heartsuit 3 < \spadesuit 3 < \cdots$$
$$< \clubsuit \mathrm{K} < \diamondsuit \mathrm{K} < \heartsuit \mathrm{K} < \spadesuit \mathrm{K} < \clubsuit \mathrm{A} < \diamondsuit \mathrm{A} < \heartsuit \mathrm{A} < \spadesuit \mathrm{A}.$$

The leftmost card of the layout indicates the suit of the hidden card. If the first card is the smallest of the four face-up cards, then the hidden card is a club (as in our ordering of the suits above); if it is the second-smallest, then the hidden card is a diamond; etc. In the example (Figure 1), the leftmost card is the smallest of the four face-up cards, so the hidden card is a club. The leftmost card is also the base for determining the rank of the hidden card.

The second card indicates whether the rank of the hidden card is above or below the rank of the leftmost card. If the second card is the smallest among the remaining face-up cards, it indicates that the hidden card's rank is between 1 and 6 lower than the first card's rank. If the second card is the highest card among the three, then it indicates that the hidden card's rank is higher than the first card. The middle face-up card indicates that the hidden card has the same rank as the first card. A circular ordering convention is used to calculate the rank of the hidden card: the rank of ace is both one higher than king and one lower than 2. In Figure 1, the 5 of clubs is the smallest among the remaining cards, so the hidden card's rank is lower than 2, the rank of the first card.

The remaining three cards form a permutation based on the ordering of the deck. There is a high (H), middle (M) and low (L) card. The permutation signifies a number from 1 to 6 according to the convention: LMH = 1, LHM = 2, MLH = 3, MHL = 4, HLM = 5, and HML = 6. For this purpose, the face-down (hidden) card, if used in the permutation, is considered higher than all 52 cards, although not assigned a rank.

In our example the last three cards form the permutation representing the number three. The rank of the hidden card is, therefore, three lower than the first card's rank (2), and thus is a queen. The magician concludes that the hidden card is the queen of clubs.

A minimum of five cards is required for such a trick, if the audience picks the hidden card. If we attempt to use four cards, then only 24 permutations of them can be formed. But there are 49 ($= 52 - 3$) possibilities for the hidden card. Therefore, it is impossible to perform a 4-card trick with the hidden card picked by the audience unless timing is used.

The assistant picks the hidden card

In this case, the audience selects n cards and the assistant chooses the hidden card. This variation makes a 4-card trick possible. Here is an example. For any given four cards, the assistant applies the following rules with top-down priority, namely, the first rule that applies is used.

1. Two or more spades are among the four cards selected.

   The assistant hides a spade chosen so that its rank is from 1 to 6 ranks higher than the highest of the remaining (face-up) spades. The hidden card is placed at the 4th position to indicate the 4th suit—spades. The three face-up cards form a permutation to indicate a number from 1 to 6. The magician adds this number to the rank of the highest face-up spade to determine the rank of the hidden spade.
2. Exactly one spade is selected.
   (a) If all four cards have the same rank x, then the assistant hides the $\clubsuit x$ and makes this special arrangement: $\diamondsuit x$ $\heartsuit x$ [$\clubsuit x$] $\spadesuit x$, where the card in square brackets is the hidden card.
   (b) If the selected cards are $\diamondsuit x$, $\heartsuit x$, $\heartsuit(x+1)$, and $\spadesuit x$, then the assistant hides $\heartsuit x$ and makes this special arrangement: $\diamondsuit x$ $\spadesuit x$ [$\heartsuit x$] $\heartsuit(x+1)$.
   (c) If the ranks of the three non-spades are from 0 to 6 (not all 0) *lower* than the spade, the assistant hides the spade at the 4th position to indicate the spades suit, and forms a permutation with the three face-up cards. The magician counts up from the lowest non-spade card to determine the rank of the hidden spade.
   (d) If the rank of one non-spade is from 1 to 6 *higher* than the spade, the assistant hides the lowest such card, and places it in its suit position—the first position is for clubs, the second for diamonds, the third for hearts, and the fourth for spades. The magician notes the permutation formed with the face-up cards, and counts up from rank of the spade to determine the rank of the hidden card.
3. No spade is selected.

   Therefore, two cards are of the same non-spade suit. The assistant hides one card of that suit, so that its rank is from 1 to 6 higher than the highest rank of the face-up cards in the suit. The placement of the hidden card indicates its suit. The magician notes the permutation formed by the face-up cards and counts up from rank of the highest card of the suit indicated by the face-down card to identify the hidden card.

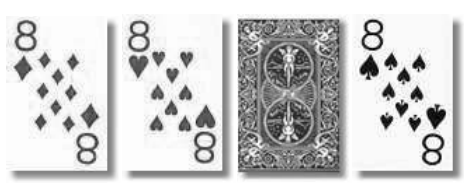

Figure 2. The hidden card is the eight of clubs.

It is very important that the first applicable rule is used, both to make and to interpret the arrangement. If Figure 2 is interpreted according to rule 2-(a), the hidden card is $\clubsuit 8$. However, if we interpret it using 2-(d), we would mistakenly conclude that the hidden card is the $\heartsuit 9$. There is another tricky case. In Figure 3, the arrangement should be interpreted using 2-(b) and we conclude that the hidden card is the ♡8. However, if the arrangement is interpreted using 2-(d), the hidden card would be mistakenly identified as the ♡9. Keep in mind that the hidden card is picked by the assistant. If the selected cards are $\diamondsuit x$, $\heartsuit x$, $\heartsuit(x+1)$, and $\spadesuit x$, we never hide the $\heartsuit(x+1)$. Other than these two cases, it should be fairly straightforward to make and to interpret an arrangement.

Figure 3. The hidden card is the eight of hearts.

This demonstrates that a 4-card trick with the hidden card picked by the assistant is possible. However, the following theorem shows that 4 is the minimum number without using timing.

Theorem 1. *A 3-card trick without timing is impossible.*

Proof. Contrary to what we want to prove, assume a 3-card trick can be accomplished. Then for every selection of 3 cards out of 52, there is a way to use 2 of these cards and a permutation (indicating a number from 1 to 6) to identify the third card (the hidden card) in the selection. Out of a standard deck of cards, there are $\binom{52}{3} = 22100$ ways to select 3 cards. On the other hand, there are only $6\binom{52}{2} = 7956$ possible combinations of 2 cards and a number from 1 to 6. Therefore, there exist 2 different 3-card selections, for which the assistant would have to use the same 2 face-up cards and the same permutation to identify the third card. Since the two 3-card selections are distinct and the 2 face-up cards are the same, the third cards have to be different. The magician would not be able to identify the hidden card. ■

Displaying cards with timing

Finally, we consider an n-card trick with timing. Here the magician watches the assistant place the cards in a row. Given a permutation of n cards, we can sequence the placement of the cards to indicate different interpretations of the same card arrangements. If we partition the n cards into k nonempty subsets, we may put down subsets of cards at k different times. This allows us to communicate more information.

The following Pascal-like triangle tabulates the number of partitions of a set of n elements into k non-empty subsets. (These are Stirling numbers of the second kind [**1**, p. 286].)

Table 1.

$n \setminus k$	0	1	2	3
0	1			
1	0	1		
2	0	1	1	
3	0	1	3	1

Using Table 1, one easily proves,

Theorem 2. *There are 78 arrangements of 3 cards in a row with timing. There are 6 arrangements of 2 cards with timing.*

Since there are more arrangements of three cards with timing than there are cards in a standard deck, one can perform a 3-card trick even allowing the audience to pick the hidden card.

Here is an example. The audience chooses three cards and picks one of them to hide. The hidden card is turned face-down. The magician appears and the assistant (with the magician watching) places the three cards in a row as follows:

The face-down card is placed first, second or third (in *time* order) to indicate that the suit of the hidden card is clubs, diamonds or hearts.

Of the two face-up cards, the assistant puts either the high card down first, to indicate the hidden card's rank is higher than 8, or puts the low card down first to indicate it is lower than 8. To indicate the hidden card is an 8, the assistant puts the two face-up cards down simultaneously. But in this way, the assistant can only identify the 8 of clubs or the 8 of diamonds. In order to identify the 8 of hearts, the assistant puts a face-up card down first and then puts the other two cards, one face-up card and one face-down card, down at the same time.

3 1 2

Figure 4. The hidden card is the 4 of clubs. The time order of placement is under each card.

To identify the spades suit, the assistant puts the face-down card and a face-up card down first at the same time. If that face-up card is the higher card of the two face-up cards, the hidden card's rank is higher than 8. Otherwise it is lower than 8. The assistant puts all three cards down at the same time to show that the hidden card is the 8 of spades.

1 2 1

Figure 5. The hidden card is the jack of spades.

Finally, the arrangement of the three cards on the table forms a permutation indicating a number from 1 to 6. This number is added or subtracted from 8 to determine the rank of the hidden card, except when it is an 8 when the permutation is ignored.

The theorem below demonstrates that a minimum of three cards is required to perform such a trick, even if the hidden card is picked by the assistant.

Theorem 3. *A 2-card trick is impossible even if the hidden card is picked by the assistant and timing is used.*

Proof. First, we observe that the assistant cannot arrange that two particular cards always be face-up, since otherwise if the audience happened to select those two cards, the assistant would be stumped. Therefore, out of the 52 card deck, each of 51 cards must be placed face-down in some arrangement.

By Theorem 2, given two cards—one face-up and one face-down—only 6 different arrangements can be made with timing. In order to name 51 face-down cards, at least $\lceil \frac{51}{6} \rceil = 9$ different cards will have to be face-up at some time. Pick 6 of these 9 cards and call them the a-cards. There are at most 36 cards that can be named using these 6 a-cards face-up. Therefore, there are at least 2 additional cards, b_1 and b_2, that can *not* be named by using an a-card face-up. If the audience selects either of the two cards, b_1 or b_2, plus an a-card, then the assistant *has* to place the b-card face-up and the a-card face-down.

In summary, for each of the b-cards, its arrangements with the six a-cards, in which it is face-up, use up all six of the possible arrangements with that b-card face-up. Therefore, if the audience selects b_1 and b_2, the assistant is stumped. ■

References

1. R. A. Brualdi, *Introductory Combinatorics*, 4th ed., Prentice-Hall, 2004.
2. M. Kleber, The best card trick, *Math. Intelligencer* **24** (2002) 9–11.
3. C. Mulcahy, Fitch Cheney's five card trick, *Math Horizons* **10** (February 2003) 8–10.

Puzzling Mechanisms, Part 3: Loony Links
M. Oskar van Deventer

The Loony Links concept was inspired by, what is called the *Boomdas* Puzzle (Figure 1). These are nine blocks connected together with dovetails. The goal of the puzzle is to solve the sliding piece puzzle without ever separating the object as a whole. The fascinating aspect of this puzzle is that all of the pieces can move anywhere, while the whole remains consistent. Unfortunately this consistency is not mechanically enforced and blocks can easily be taken off.

I wanted a mechanism that forces the pieces to stay together as a whole, while retaining the freedom of motion of the individual pieces. A minimum of three pieces would be needed, so I set myself the goal of designing a chain of three links with the property that any link could switch places with any other link, without the chain as a whole being able to come apart.

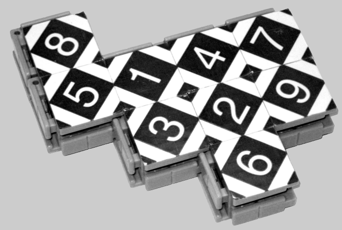

Figure 1. The *Boomdas* Puzzle.

At first, I built some prototypes of layered cardboard. That was a lot of work. Unfortunately, while allowing me to demonstrate the concept, all cardboard prototypes (see Figure 2) could easily be taken apart.

Figure 2. A cardboard prototype of the Loony Links concept.

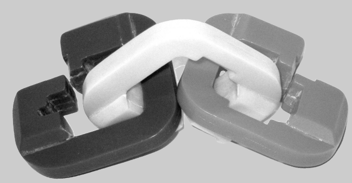

Figure 3. Linked Hearts, the result of many plastic prototypes.

Solid polystyrene proved to be a better material to make prototypes, and I succeeded in making a working sample which I called *Linked Hearts* (Figure 3). The mechanism works well, but with the flexibility of the plastic it is still too easy to cheat.

At some point in time I learned how to cast tin, which proved to be a good material for this puzzle. After destroying many wooden moulds and reusing the tin, I finally had a satisfactory prototype. I used an electrolysis set to gold- and copper-plate some of the pieces. The *Gold-Silver-Bronze* puzzle (Figure 4) received an Honorable Mention in the 2002 International Puzzle Design Competition.

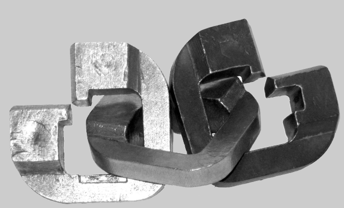

Figure 4. Gold-Silver-Bronze, the result of many tin prototypes.

Figure 5. *Chain* as perfected by Noji Kitajima and produced by Hanayama.

At this point, Nob Yoshigaraha picked up the puzzle for Hanayama and commissioned Noji Kitajima to perfect the puzzle, which became the *Chain* puzzle (Figure 5). This puzzle has been made with such incredible accuracy that even I can not solve it without checking Hanayama's instructions. It is possible to combine many *Chain* puzzles and make one long chain. Unfortunately, the puzzle then looses its property that the links can be interchanged.

In parallel with Hanayama, James Dalgety produced the *GGG* puzzle (Figure 6). My original *Linked Hearts* design was made as a mechanical maze where some of the moves of the puzzle were blocked as in a maze. It took James' maker many attempts to perfect this puzzle, which is even more difficult than the already almost unsolvable Hanayama Chain.

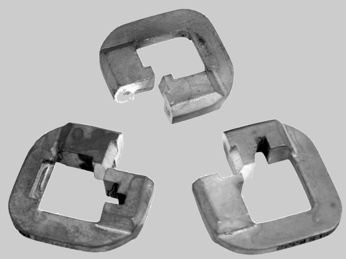

Figure 6. *GGG* or *One-and-a-Half-Horse*, produced by James Dalgety.

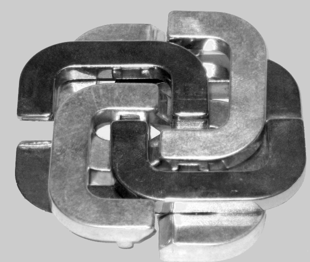

Figure 7. *Lucky Clover*, produced by Bits & Pieces.

So, after many hundreds of hours work and many tens of prototypes over a period of almost a decade, the Loony Links mechanism was finally implemented successfully. My next dream is to expand the concept to four or more links that can be swapped without the chain coming apart. So far, all my attempts in this direction have failed miserably. The closest that I have come is *Lucky Clover*, a puzzle consisting of four identical links which requires each pair to switch parity before the puzzle can come apart (Figure 7).

[*Puzzling Mechanisms* continues on page 211.]

Family Occasion

"It was a wonderful party," said Lucilla to her friend Harriet.

"Who was there?"

"Well—there was one grandfather, one grandmother, two fathers, two mothers, four children, three grandchildren, one brother, two sisters, two sons, two daughters, one father- in-law, one mother-in-law and one daughter-in-law."

"Wow! Twenty-three people!"

"No, it was less than that. A lot less."

What is the *smallest* size of a party that is consistent with Lucilla's description?

—from *Professor Stewart's Cabinet of Mathematical Curiosities*, by Ian Stewart (p. 105). Reviewed on page 223.

(The solution is on page 227.)