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FACTORIZATIONS OF SOME PERIODIC LINEAR RECURRENCE SYSTEMS

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Abstract. Let P and Q be relatively prime integers. The Lucas sequences are defined by $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = P$, and

$$U_n = PU_{n-1} - QU_{n-2} \quad \text{and} \quad V_n = PV_{n-1} - QV_{n-2},$$

where $n \geq 2$. We show that

$$U_n = \prod_{k=1}^{n-1} \left(P - 2\sqrt{Q} \cos \frac{k\pi}{n} \right), \qquad n \geq 2,$$

$$V_n = \prod_{k=1}^{n} \left(P - 2\sqrt{Q} \cos \frac{(k-1/2)\pi}{n} \right), \qquad n \geq 1.$$

The proofs depend on finding the eigenvalues and eigenvectors of certain tridiagonal matrices.

Next, let a_1, a_2, b_1, and b_2 be real numbers. A period two second order linear recurrence system is defined to be the sequence $f_0 = 1$, $f_1 = a_1$, and

$$f_{2n} = a_2 f_{2n-1} + b_1 f_{2n-2} \quad \text{and} \quad f_{2n+1} = a_1 f_{2n} + b_2 f_{2n-1}$$

for $n \geq 1$. Also, let $D = a_1 a_2 + b_1 + b_2$ and assume $D^2 - 4b_1 b_2 \neq 0$. We show that

$$f_{2n+1} = a_1 \prod_{k=1}^{n} \left(\frac{a_1 + a_2}{2} \pm \sqrt{\left(\frac{a_1 - a_2}{2} \right)^2 - b_1 - b_2 + 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}} \right)$$

for $n \geq 0$. The proof also depends on finding the eigenvalues and eigenvectors of a certain tridiagonal matrix.

1. Introduction. Complex factorizations of integer sequences show connections between complex numbers and integers and demonstrate properties of these sequences. In this paper, we find factorization formulas for terms of Lucas sequences and for the odd terms of a period two second order linear recurrence system. To derive these formulas we determine the eigenvalues of

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certain tridiagonal matrices. This gives an application of linear algebra to number theory.

We begin our discussion with the definition of Lucas sequences.

DEFINITION 1.1. Let P and Q be relatively prime integers. The *Lucas sequences* are defined by $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = P$, and

$$U_n = PU_{n-1} - QU_{n-2} \quad \text{and} \quad V_n = PV_{n-1} - QV_{n-2},$$

where $n \geq 2$.

These sequences were originally studied by Lucas [3]. The Fibonacci and Lucas numbers, F_n and L_n, are special cases of the U and V sequences when $P = 1$ and $Q = -1$.

Next, we define a period k second order linear recurrence system. This definition is related to the system defined in [2, p. 440, Theorem C1]. The term "period k" refers to the k terms in the sequences a and b.

DEFINITION 1.2. Let k be a positive integer. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be real numbers. A *period k second order linear recurrence system* is defined to be the sequence $f_0 = 1$, $f_2 = a_1$, and

$$\begin{aligned} f_{kn+2} &= a_2 f_{kn+1} + b_1 f_{kn}, \\ f_{kn+3} &= a_3 f_{kn+2} + b_2 f_{kn+1}, \\ \vdots &= \vdots \\ f_{kn+k} &= a_k f_{kn+k-1} + b_{k-1} f_{kn+k-2}, \\ f_{kn+k+1} &= a_1 f_{kn+k} + b_k f_{kn+k-1}, \end{aligned}$$

for $n \geq 0$.

2. Factorization of Lucas sequences. We find factorization formulas for the Lucas sequences. That is, we find a product formula for U_n and V_n. This generalizes a result by Cahill, D'Errico, and Spence [1] where they found complex factorizations of the Fibonacci and Lucas numbers. In particular, they showed that

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i \cos \frac{k\pi}{n}\right), \qquad n \geq 2,$$

$$L_n = \prod_{k=1}^{n} \left(1 - 2i \cos \frac{(k-1/2)\pi}{n}\right), \quad n \geq 1.$$

Here, $i = \sqrt{-1}$.

In addition to finding complex factorization formulas for U and V, our proofs determine the eigenvalues and eigenvectors of certain tridiagonal matrices.

It should be noted that for $n \geq 0$, Stewart [5, pp. 88–89] found factorization formulas for the nth cyclotomic polynomials of two variables. That is, for real numbers r_1 and r_2, Stewart found factorizations for the nth cyclotomic polynomial $\Phi_n(r_1, r_2)$ in terms of the primitive roots of unity. There is a connection between Stewart's factorizations of $\Phi_n(r_1, r_2)$ and our factorization formula for U_n. To see this, we begin by denoting the roots of the polynomial $x^2 - Px + Q$ by

$$r_1, r_2 = \frac{P \pm \sqrt{P^2 - 4Q}}{2}.$$

If $P^2 - 4Q \neq 0$, from [4, p. 316, eq. (2)] we have

$$U_n = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

And from [5, p. 88],

$$r_1^n - r_2^n = \prod_{d|n} \Phi_d(r_1, r_2).$$

Combining these results with Stewart's factorizations for $\Phi_n(r_1, r_2)$, we have a factorization formula for U_n.

We begin with the following lemma which generalizes a result in [2, p. 443, Theorem D1].

LEMMA 2.1. *Let n be a positive integer and let a, b, and c be real numbers. Let $T(n)$ be the $n \times n$ tridiagonal matrix*

$$T(n) = \begin{pmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{pmatrix}.$$

Then the eigenvalues of $T(n)$ are

$$a - 2\sqrt{bc}\cos\frac{k\pi}{n+1}, \qquad 1 \leq k \leq n.$$

In addition, an eigenvector corresponding to the kth eigenvalue is

$$x_k = \begin{pmatrix} \sin\frac{k\pi}{n+1} \\ -\frac{\sqrt{bc}}{b}\sin\frac{2k\pi}{n+1} \\ \left(-\frac{\sqrt{bc}}{b}\right)^2\sin\frac{3k\pi}{n+1} \\ \vdots \\ \left(-\frac{\sqrt{bc}}{b}\right)^{j-1}\sin\frac{jk\pi}{n+1} \\ \vdots \\ \left(-\frac{\sqrt{bc}}{b}\right)^{n-1}\sin\frac{nk\pi}{n+1} \end{pmatrix}.$$

Proof. We examine $T(n)x_k$ and $\left(a - 2\sqrt{bc}\cos\frac{k\pi}{n+1}\right)x_k$. The first component of $T(n)x_k$ is

$$a\sin\frac{k\pi}{n+1} + b\left(-\frac{\sqrt{bc}}{b}\sin\frac{2k\pi}{n+1}\right) = a\sin\frac{k\pi}{n+1} - \sqrt{bc}\sin\frac{2k\pi}{n+1}$$

and the first component of $\left(a - 2\sqrt{bc}\cos\frac{k\pi}{n+1}\right)x_k$ is

$$a\sin\frac{k\pi}{n+1} - 2\sqrt{bc}\cos\frac{k\pi}{n+1}\sin\frac{k\pi}{n+1}.$$

Since

$$\sin\frac{2k\pi}{n+1} = 2\sin\frac{k\pi}{n+1}\cos\frac{k\pi}{n+1},$$

these two components are equal.

The jth component of $T(n)x_k$, for $1 < j < n$, is

$$c\left(-\frac{\sqrt{bc}}{b}\right)^{j-2}\sin\frac{(j-1)k\pi}{n+1} + a\left(-\frac{\sqrt{bc}}{b}\right)^{j-1}\sin\frac{jk\pi}{n+1} + b\left(-\frac{\sqrt{bc}}{b}\right)^{j}\sin\frac{(j+1)k\pi}{n+1}. \tag{2.1}$$

The jth component of $\left(a - 2\sqrt{bc}\cos\frac{k\pi}{n+1}\right)x_k$ is

$$a\left(-\frac{\sqrt{bc}}{b}\right)^{j-1}\sin\frac{jk\pi}{n+1} - 2\sqrt{bc}\left(-\frac{\sqrt{bc}}{b}\right)^{j-1}\cos\frac{k\pi}{n+1}\sin\frac{jk\pi}{n+1}. \tag{2.2}$$

We want to show that (2.1) equals (2.2). First, the term

$$a\left(-\frac{\sqrt{bc}}{b}\right)^{j-1}\sin\frac{jk\pi}{n+1}$$

is in both (2.1) and (2.2). Using the trigonometric identity

$$2\sin A\cos B = \sin(A+B) + \sin(A-B), \tag{2.3}$$

we see that the last term in (2.2) is

$$-2\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-1}\sin\frac{jk\pi}{n+1}\cos\frac{k\pi}{n+1}$$
$$= -\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-1}\sin\frac{(j+1)k\pi}{n+1} - \sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-1}\sin\frac{(j-1)k\pi}{n+1}.$$

Since

$$-\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-1} = c\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-2}$$

and

$$\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-1} = b\Big(-\frac{\sqrt{bc}}{b}\Big)^{j},$$

we obtain

$$c\Big(-\frac{\sqrt{bc}}{b}\Big)^{j-2}\sin\frac{(j-1)k\pi}{n+1} + b\Big(-\frac{\sqrt{bc}}{b}\Big)^{j}\sin\frac{(j+1)k\pi}{n+1}.$$

Thus, the remaining terms in (2.1) give the last term in (2.2). This completes the case of the jth component for $1 < j < n$.

Finally, the nth component of $T(n)x_k$ is

$$c\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-2}\sin\frac{(n-1)k\pi}{n+1} + a\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{nk\pi}{n+1} \tag{2.4}$$

and the nth component of $\big(a - 2\sqrt{bc}\cos\frac{k\pi}{n+1}\big)x_k$ is

$$a\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{nk\pi}{n+1} - 2\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\cos\frac{k\pi}{n+1}\sin\frac{nk\pi}{n+1}. \tag{2.5}$$

We want to show that (2.4) equals (2.5). First, the term

$$a\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{nk\pi}{n+1}$$

is in both (2.4) and (2.5). Using (2.3), we find that the last term in (2.5) is

$$-2\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{nk\pi}{n+1}\cos\frac{k\pi}{n+1}$$
$$= -\sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{(n+1)k\pi}{n+1} - \sqrt{bc}\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-1}\sin\frac{(n-1)k\pi}{n+1}$$
$$= c\Big(-\frac{\sqrt{bc}}{b}\Big)^{n-2}\sin\frac{(n-1)k\pi}{n+1}.$$

Thus, the remaining term in (2.5) is equal to the remaining term in (2.4). The result follows. ∎

We now state and prove a theorem for U_n.

THEOREM 2.2. *Let $n \geq 2$ be a positive integer. Then*

$$U_n = \prod_{k=1}^{n-1} \left(P - 2\sqrt{Q} \cos \frac{k\pi}{n} \right).$$

Proof. Let $M(n-1)$ be the $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} P & \sqrt{Q} & & & & \\ \sqrt{Q} & P & \sqrt{Q} & & & \\ & \sqrt{Q} & P & \sqrt{Q} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{Q} & P & \sqrt{Q} \\ & & & & \sqrt{Q} & P \end{pmatrix}.$$

We note that $U_n = \det M(n-1)$ for $n \geq 2$. Also, the determinant of a matrix can be found by taking the product of its eigenvalues. By Lemma 2.1, the eigenvalues of $M(n-1)$ are

$$P - 2\sqrt{Q} \cos \frac{k\pi}{n},$$

where $1 \leq k \leq n-1$. The result follows. ∎

Next, we need the following lemma to find a product formula for V_n.

LEMMA 2.3. *Let n be a positive integer and let a and b be real numbers. Let $R(n)$ be the $n \times n$ tridiagonal matrix*

$$R(n) = \begin{pmatrix} a & 2b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & b & a & b \\ & & & & b & a \end{pmatrix}.$$

Then the eigenvalues of $R(n)$ are

$$a - 2b \cos \frac{(k-1/2)\pi}{n}, \qquad 1 \leq k \leq n.$$

In addition, an eigenvector corresponding to the kth eigenvalue is

$$x_k = \begin{pmatrix} \sin\frac{(k-1/2)\pi}{n} \\ -\sin\frac{(k-1/2)\pi}{n}\cos\frac{(k-1/2)\pi}{n} \\ \sin\frac{(k-1/2)\pi}{n}\cos\frac{2(k-1/2)\pi}{n} \\ \vdots \\ (-1)^{j-1}\sin\frac{(k-1/2)\pi}{n}\cos\frac{(j-1)(k-1/2)\pi}{n} \\ \vdots \\ (-1)^{n-1}\sin\frac{(k-1/2)\pi}{n}\cos\frac{(n-1)(k-1/2)\pi}{n} \end{pmatrix}.$$

Proof. To simplify the notation in the proof, let

$$\theta = \frac{(k-1/2)\pi}{n}.$$

We examine $R(n)x_k$ and $(a - 2b\cos\theta)x_k$. The first component of $R(n)x_k$ is

$$a\sin\theta - 2b\sin\theta\cos\theta$$

and the first component of $(a - 2b\cos\theta)x_k$ is

$$a\sin\theta - 2b\cos\theta\sin\theta.$$

These two components are equal.

The jth component of $R(n)x_k$, for $1 < j < n$, is

$$(-1)^{j-2}b\sin\theta\cos((j-2)\theta) + (-1)^{j-1}a\sin\theta\cos((j-1)\theta) + (-1)^{j}b\sin\theta\cos(j\theta).$$

The jth component of $(a - 2b\cos\theta)x_k$ is

$$(-1)^{j-1}a\sin\cos((j-1)\theta) + (-1)^{j}2b\sin\theta\cos\theta\cos((j-1)\theta).$$

First, the term

$$(-1)^{j-1}a\sin\theta\cos((j-1)\theta)$$

is in both expressions. Next, the signs in the remaining terms are the same and each term contains a b and a $\sin\theta$. Thus, it suffices to show

$$\cos((j-2)\theta) + \cos(j\theta) = 2\cos((j-1)\theta)\cos\theta.$$

However, this equation is just a special case of the trigonometric identity

$$2\cos A\cos B = \cos(A+B) + \cos(A-B). \tag{2.6}$$

This completes the proof for the jth component, $1 < j < n$.

Finally, the nth component of $R(n)x_k$ is

$$(-1)^{n-2}b\sin\theta\cos((n-2)\theta) + (-1)^{n-1}a\sin\theta\cos((n-1)\theta)$$

and the nth component of $(a - 2b\cos\theta)x_k$ is

$$(-1)^{n-1}a\sin\theta\cos((n-1)\theta) + (-1)^{n}2b\sin\theta\cos\theta\cos((n-1)\theta).$$

Both components contain the term

$$(-1)^{n-1}a\sin\theta\cos((n-1)\theta).$$

The remaining terms have the same sign and both contain a b and a $\sin\theta$. Thus, it suffices to show

$$\cos((n-2)\theta) = 2\cos((n-1)\theta)\cos\theta.$$

Using (2.6), we obtain

$$2\cos((n-1)\theta)\cos\theta = \cos(n\theta) + \cos((n-2)\theta).$$

Since $\cos(n\theta) = 0$, the result follows. ■

Next, we state and prove a theorem for the sequence V_n.

THEOREM 2.4. *Let n be a positive integer. Then*

$$V_n = \prod_{k=1}^{n}\left(P - 2\sqrt{Q}\cos\frac{(k-1/2)\pi}{n}\right).$$

Proof. Let $S(n)$ be the $n \times n$ matrix

$$\begin{pmatrix} P & 2\sqrt{Q} & & & & \\ \sqrt{Q} & P & \sqrt{Q} & & & \\ & \sqrt{Q} & P & \sqrt{Q} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{Q} & P & \sqrt{Q} \\ & & & & \sqrt{Q} & P \end{pmatrix}.$$

We note that $V_n = \det S(n)$ for $n \geq 1$. Also, the determinant of a matrix can be found by taking the product of its eigenvalues. By Lemma 2.3, the eigenvalues of $S(n)$ are

$$P - 2\sqrt{Q}\cos\frac{(k-1/2)\pi}{n},$$

where $1 \leq k \leq n$. The result follows. ■

3. Factorization of a period two second order linear recurrence system. We now proceed to find a factorization of a period two second order linear recurrence system. The next lemmas help us do this. Part of the proofs of these results can be found in [2]. Because the proofs in [2] omit several details and we generalize the results, we supply detailed proofs. The first lemma, Lemma 3.1, gives a Binet-like formula for f_n.

LEMMA 3.1. *Let $\{f_n \mid n = 0, 1, 2, \ldots\}$ be a period two second order linear recurrence system. Let $D = a_1a_2 + b_1 + b_2$,*

$$\alpha = \sqrt{\frac{D + \sqrt{D^2 - 4b_1b_2}}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{D - \sqrt{D^2 - 4b_1b_2}}{2}}.$$

Assume $D^2 - 4b_1b_2 \neq 0$ *and let*

$$A(\alpha,\beta) = \frac{\alpha^2 + a_1\alpha - b_2}{2(\alpha^2 - \beta^2)}.$$

Let n *be a nonnegative integer. Then*

$$f_n = A(\alpha,\beta)\alpha^n + A(-\alpha,\beta)(-\alpha)^n + A(\beta,\alpha)\beta^n + A(-\beta,\alpha)(-\beta)^n.$$

Proof. Let

$$G(t) = \sum_{n\geq 0} f_n t^n.$$

Also, let

$$G_1(t) = \sum_{n\geq 0} f_{2n+1}t^{2n+1} \quad \text{and} \quad G_2(t) = \sum_{n\geq 0} f_{2n}t^{2n}.$$

Then

$$\begin{aligned} f_{2n}t^{2n} &= a_2 t f_{2n-1}t^{2n-1} + b_1 t^2 f_{2n-2}t^{2n-2}, \\ f_{2n+1}t^{2n+1} &= a_1 t f_{2n}t^{2n} + b_2 t^2 f_{2n-1}t^{2n-1} \end{aligned}$$

for $n \geq 1$. Hence,

$$\begin{aligned} G_2(t) - 1 &= a_2 t G_1(t) + b_1 t^2 G_2(t), \\ G_1(t) - a_1 t &= a_1 t(G_2(t) - 1) + b_2 t^2 G_1(t). \end{aligned}$$

Putting this in matrix form, we have

$$\begin{pmatrix} -a_2 t & 1 - b_1 t^2 \\ 1 - b_2 t^2 & -a_1 t \end{pmatrix} \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solving this system for $G_1(t)$ and $G_2(t)$ by Cramer's rule, we get

$$\begin{aligned} G_1(t) &= \frac{-a_1 t}{-1 + (b_1 + b_2 + a_1a_2)t^2 - b_1b_2t^4}, \\ G_2(t) &= \frac{b_2t^2 - 1}{-1 + (a_1a_2 + b_1 + b_2)t^2 - b_1b_2t^4}. \end{aligned}$$

Therefore,

$$G(t) = G_1(t) + G_2(t) = \frac{1 + a_1 t - b_2 t^2}{1 - (b_1 + b_2 + a_1a_2)t^2 + b_1b_2t^4}.$$

Factoring, we obtain

$$1 - Dt^2 + b_1b_2t^4 = (1 - \alpha^2t^2)(1 - \beta^2t^2) = (1 - \alpha t)(1 + \alpha t)(1 - \beta t)(1 + \beta t).$$

Since $D^2 - 4b_1b_2 \neq 0$, $\alpha \neq \beta$. Using partial fraction decomposition, we want to find A_1, A_2, B_1, and B_2 such that

$$\frac{1 + a_1 t - b_2 t}{1 - Dt^2 + b_1b_2t^4} = \frac{A_1}{1 - \alpha t} + \frac{A_2}{1 + \alpha t} + \frac{B_1}{1 - \beta t} + \frac{B_2}{1 + \beta t}.$$

But

$$A_1 = A(\alpha, \beta).$$

Also, $A_2 = A(-\alpha, \beta)$, $A_3 = A(\beta, \alpha)$, and $A_4 = A(-\beta, \alpha)$. Thus,

$$f_n = A(\alpha,\beta)\alpha^n + A(-\alpha,\beta)(-\alpha)^n + A(\beta,\alpha)\beta^n + A(-\beta,\alpha)(-\beta)^n$$

for any nonnegative integer n. ■

Definition 3.2. Let n be a positive integer and let $T(n)$ be the $n \times n$ tridiagonal matrix consisting of the upper left n rows and n columns of the infinite matrix

$$T = \begin{pmatrix} a_1 & b_1 & & & & & & \\ -1 & a_2 & b_2 & & & & & \\ & -1 & a_1 & b_1 & & & & \\ & & -1 & a_2 & b_2 & & & \\ & & & -1 & a_1 & b_1 & & \\ & & & & -1 & a_2 & b_2 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where a_1, a_2, b_1, and b_2 are real numbers.

Lemma 3.3. *Let n be a nonnegative integer. Then* $\det T(2n+1) = 0$ *if and only if*

$$a_1 = 0 \quad or \quad b_1 + b_2 + a_1 a_2 = 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}$$

for some $1 \le k \le n$.

Proof. We first note that if $\det T(2n+1) = 0$, then a_1 could be zero. And if $a_1 = 0$, then $\det T(2n+1) = 0$. Next, let $f_n = \det T(n)$. Then $\{f_n \mid n = 0, 1, 2, \ldots\}$ is a period two second order linear recurrence system. Therefore, by Lemma 3.1, $f_{2n+1} = \det T(2n+1) = 0$ if and only if

$$A(\alpha,\beta)\alpha^{2n+1} + A(-\alpha,\beta)(-\alpha)^{2n+1} + A(\beta,\alpha)\beta^{2n+1} + A(-\beta,\alpha)(-\beta)^{2n+1} = 0.$$

Since $(-1)^{2n+1} = -1$, we have

$$\alpha^{2n+1}(A(\alpha,\beta) - A(-\alpha,\beta)) + \beta^{2n+1}(A(\beta,\alpha) - A(-\beta,\alpha)) = 0.$$

But

$$\frac{A(\alpha,\beta) - A(-\alpha,\beta)}{A(\beta,\alpha) - A(-\beta,\alpha)} = -\frac{\alpha}{\beta},$$

so

$$\begin{aligned}\alpha^{2n+1}&(A(\alpha,\beta)-A(-\alpha,\beta))+\beta^{2n+1}(A(\beta,\alpha)-A(-\beta,\alpha))\\ &=\alpha^{2n+1}\left(-\frac{\alpha}{\beta}\right)(A(\beta,\alpha)-A(-\beta,\alpha))+\beta^{2n+1}(A(\beta,\alpha)-A(-\beta,\alpha))\\ &=\left(\left(-\frac{\alpha^{2n+2}}{\beta}\right)+\beta^{2n+1}\right)(A(\beta,\alpha)-A(-\beta,\alpha))=0.\end{aligned}$$

Thus,

$$-\alpha^{2n+2}+\beta^{2n+2}=0,\quad \text{i.e.}\quad \frac{\alpha^{2n+2}}{\beta^{2n+2}}=1.$$

Hence, for some $0\le k\le n$, we have

$$\frac{\alpha^{2n+2}}{\beta^{2n+2}}=e^{2k\pi i}.$$

Let

$$\theta=\frac{2k\pi}{n+1}$$

for some $0\le k\le n$. Then

$$\frac{D+\sqrt{D^2-4b_1b_2}}{D-\sqrt{D^2-4b_1b_2}}=\frac{\alpha^2}{\beta^2}=e^{i\theta}.$$

Simplifying, we have

$$D+\sqrt{D^2-4b_1b_2}=(D-\sqrt{D^2-4b_1b_2})e^{i\theta}.$$

We note here that $k\neq 0$ since $D^2-4b_1b_2\neq 0$. Next,

$$\sqrt{D^2-4b_1b_2}\,e^{i\theta}+\sqrt{D^2-4b_1b_2}=De^{i\theta}-D.$$

So,

$$\sqrt{D^2-4b_1b_2}\,(e^{i\theta}+1)=D(e^{i\theta}-1).$$

Hence,

$$\sqrt{D^2-4b_1b_2}=D\frac{e^{i\theta}-1}{e^{i\theta}+1}=D\frac{e^{i\theta}-1}{e^{i\theta}+1}\cdot\frac{e^{-i\theta}+1}{e^{-i\theta}+1}=D\frac{e^{i\theta}-e^{-i\theta}}{2+e^{i\theta}+e^{-i\theta}}.$$

Now since

$$e^{i\theta}=\cos\theta+i\sin\theta$$

and

$$\sin(-\theta)=-\sin\theta\quad\text{and}\quad\cos(-\theta)=\cos\theta,$$

we have

$$D\frac{e^{i\theta}-e^{-i\theta}}{2+e^{i\theta}+e^{-i\theta}}=D\frac{i\sin\theta}{1+\cos\theta}=Di\tan\frac{\theta}{2}.$$

Squaring both sides of the equality, we have

$$D^2-4b_1b_2=-D^2\tan^2\frac{\theta}{2}=-D^2\left(\sec^2\frac{\theta}{2}-1\right).$$

Thus,

$$4b_1b_2 = D^2 \sec^2 \frac{\theta}{2}, \quad \text{so} \quad D^2 = 4b_1b_2 \cos^2 \frac{\theta}{2},$$

and therefore

$$D = 2\sqrt{b_1b_2} \cos \frac{\theta}{2}.$$

Substituting for D and θ, we get

$$b_1 + b_2 + a_1a_2 = 2\sqrt{b_1b_2} \cos \frac{k\pi}{n+1}$$

for some $1 \le k \le n$. This is what we wanted to prove. ∎

LEMMA 3.4. *Let n be a nonnegative integer. The eigenvalues of $T(2n+1)$ are*

$$a_1 \quad \text{and} \quad \frac{a_1 + a_2}{2} \pm \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 - b_1 - b_2 + 2\sqrt{b_1b_2} \cos \frac{k\pi}{n+1}},$$

$1 \le k \le n$. *In addition, an eigenvector corresponding to the eigenvalue λ is*

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{2n} \end{pmatrix},$$

where

$$\begin{aligned}
x_0 &= b_1^n b_2^n p_0(\lambda), \\
x_1 &= b_1^{n-1} b_2^n (-p_1(\lambda)), \\
x_2 &= b_1^{n-1} b_2^{n-1} p_2(\lambda), \\
x_3 &= b_1^{n-2} b_2^{n-1} (-p_3(\lambda)), \\
x_4 &= b_1^{n-2} b_2^{n-2} p_4(\lambda), \\
\vdots &= \vdots \\
x_{2j-1} &= b_1^{n-j} b_2^{n-j+1} (-p_{2j-1}(\lambda)), \\
x_{2j} &= b_1^{n-j} b_2^{n-j} p_{2j}(\lambda), \\
\vdots &= \vdots \\
x_{2n} &= p_{2n}(\lambda)
\end{aligned}$$

and for $0 \le j \le 2n$, $p_j(x)$ is the characteristic polynomial of the matrix consisting of the upper right j rows and j columns of T.

Proof. Let $g_0 = 1$, $g_1 = a_1 - x$, and

$$g_{2n} = (a_2 - x)g_{2n-1} + b_1 g_{2n-2}, \qquad g_{2n+1} = (a_1 - x)g_{2n} + b_2 g_{2n-1}$$

for $n \geq 1$. The eigenvalues of $T(2n+1)$ are the solutions of $\det T(2n+1) = g_{2n+1} = 0$. By Lemma 3.3, this implies $a_1 - x = 0$ or, for some $1 \leq k \leq n$,

$$b_1 + b_2 + (a_1 - x)(a_2 - x) = 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}.$$

Therefore, the eigenvalues of $T(2n+1)$ are a_1 and the solutions of the quadratic equation

$$x^2 - (a_1 + a_2)x + a_1 a_2 + b_1 + b_2 = 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}$$

for some $1 \leq k \leq n$. Completing the square, we have

$$x^2 - (a_1 + a_2)x + \left(\frac{a_1 + a_2}{2}\right)^2 = \left(\frac{a_1 + a_2}{2}\right)^2 - a_1 a_2 - b_1 - b_2 + 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}.$$

Therefore, the eigenvalues of $T(2n+1)$ are a_1 and

$$\frac{a_1 + a_2}{2} \pm \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 - b_1 - b_2 + 2\sqrt{b_1 b_2} \cos \frac{k\pi}{n+1}}$$

for some $1 \leq k \leq n$. Now, to verify the eigenvector x for the corresponding eigenvalue λ, we check each component of $T(2n+1)x$ and λx.

The 0th component of $T(2n+1)x$ is

$$a_1 b_1^n b_2^n p_0(\lambda) + b_1 b_1^{n-1} b_2^n(-p_1(\lambda))$$

and the 0th component of λx is

$$\lambda b_1^n b_2^n p_0(\lambda).$$

But

$$p_1(\lambda) = a_1 - \lambda \quad \text{and} \quad p_0(\lambda) = 1$$

so the 0th components are equal.

The first components of $T(2n+1)x$ and λx are

$$-b_1^n b_2^n p_0(\lambda) + a_2 b_1^{n-1} b_2^n(-p_1(\lambda)) + b_2 b_1^{n-1} b_2^{n-1} p_2(\lambda), \qquad \lambda b_1^{n-1} b_2^n(-p_1(\lambda)),$$

respectively. By the fact that $p_2(\lambda) = (a_2 - \lambda)(a_1 - \lambda) + b_1$ and simplifying, it follows that the powers of λ in both components are equal, so the first components are equal.

For $1 \leq j < n$, the $2j$th components of $T(2n+1)x$ and λx are

$$(-1)\big(b_1^{n-j} b_2^{n-j+1}(-p_{2j-1}(\lambda))\big) + a_1\big(b_1^{n-j} b_2^{n-j} p_{2j}(\lambda)\big) + b_1\big(b_1^{n-j-1} b_2^{n-j}(-p_{2j+1}(\lambda))\big)$$

and

$$\lambda b_1^{n-j} b_2^{n-j} p_{2j}(\lambda),$$

respectively. But $b_1(b_1^{n-j-1}b_2^{n-j}p_{2j+1}(\lambda))$ is $b_1^{n-j}b_2^{n-j}$ times the characteristic polynomial of the upper left $2j+1$ rows and columns of T. Via expansion by minors, this product is $b_1^{n-j}b_2^{n-j}$ times

$$(a_1-\lambda)p_{2j}(\lambda)+b_2p_{2j-1}(\lambda).$$

Thus, the $2j$th components are equal.

For $1 \le j \le n$, the $(2j-1)$th components of $T(2n+1)x$ and λx are

$$(-1)\big(b_1^{n-j+1}b_2^{n-j+1}p_{2j-2}(\lambda)\big)+a_2(b_1^{n-j}b_2^{n-j+1}(-p_{2j-1}(\lambda))) \\ +b_2\big(b_1^{n-j}b_2^{n-j}p_{2j}(\lambda)\big)$$

and

$$\lambda b_1^{n-j}b_2^{n-j+1}(-p_{2j-1}(\lambda)),$$

respectively. But $b_2(b_1^{n-j}b_2^{n-j})p_{2j}(\lambda)$ is $b_1^{n-j}b_2^{n-j+1}$ times the characteristic polynomials of the upper left $2j$ rows and columns of T. Via expansion by minors, this product is $b_1^{n-j}b_2^{n-j+1}$ times

$$(a_2-\lambda)p_{2j-1}(\lambda)+b_1p_{2j-2}(\lambda).$$

Thus, the $(2j-1)$th components are equal.

Finally, the $2n$th components of $T(2n+1)x$ and λx are

$$(-1)\big(b_2(-p_{2n-1}(\lambda))\big)+a_1p_{2n}(\lambda) \quad \text{and} \quad \lambda p_{2n}(\lambda),$$

respectively. But these two components are equal since λ is an eigenvalue of $T(2n+1)$ and the characteristic polynomial of $T(2n+1)$, evaluated at λ, is

$$(a_1-\lambda)p_{2n}(\lambda)+b_2p_{2n-1}(\lambda).$$

The result follows. ∎

THEOREM 3.5. *Let $\{f_n \mid n=0,1,2,\ldots\}$ be a period two second order linear recurrence system. Let n be a nonnegative integer. Then*

$$f_{2n+1}=a_1\prod_{k=1}^{n}\left(\frac{a_1+a_2}{2}\pm\sqrt{\left(\frac{a_1-a_2}{2}\right)^2-b_1-b_2+2\sqrt{b_1b_2}\cos\frac{k\pi}{n+1}}\right).$$

Proof. The result follows from Lemma 3.4, $f_{2n+1}=\det T(2n+1)$, and the fact that the determinant of a matrix is the product of its eigenvalues. ∎

4. Open questions. There are several open questions for future work. The authors believe that the methods and proof techniques used in this paper are applicable to higher order sequences and larger period systems, as well as non-Lucas sequences, although the results and computations may be more complicated.

1. Find a factorization formula for a second order linear recurrence with general initial conditions, i.e., $G_0=a$, $G_1=b$, and for $n\ge 2$, $G_n=$

$PG_{n-1} - QG_{n-2}$. It should be noted that the upper left n rows and n columns of the infinite tridiagonal matrix

$$S = \begin{pmatrix} b & a\sqrt{Q} & & & \\ \sqrt{Q} & P & \sqrt{Q} & & \\ & \sqrt{Q} & P & \sqrt{Q} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

have the property that for $n \geq 1$, $G_n = \det S(n)$.

2. Determine factorization formulas for higher order linear recurrences.
3. What are the eigenvalues and eigenvectors of $T(2n)$? Find a factorization formula for f_{2n}, the even terms in a period two second order linear recurrence system.
4. Find a factorization formula for a period three second order recurrence system. Let n be a positive integer and let $W(n)$ be the $n \times n$ tridiagonal matrix consisting of the upper left n rows and n columns of the infinite matrix

$$W = \begin{pmatrix} a_1 & b_1 & & & & & & \\ -1 & a_2 & b_2 & & & & & \\ & -1 & a_3 & b_3 & & & & \\ & & -1 & a_1 & b_1 & & & \\ & & & -1 & a_2 & b_2 & & \\ & & & & -1 & a_3 & b_3 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where a_1, a_2, a_3, b_1, b_2, and b_3 are real numbers. What are the eigenvalues and eigenvectors of the tridiagonal matrix $T(3n+1)$?
5. Find a factorization formula for a period k second order recurrence system. Finally, find a factorization formula for a period k mth order recurrence system.

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