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The Chinese Ring puzzle, the Crazy Elephant Dance puzzle, the *b*-Spinout puzzle, and Gray codes

CURTIS COOPER

1. *Chinese ring puzzle and the Spinout puzzle*

The Chinese Ring puzzle consists of a long loop with a handle on one end and nine rings entwined on the loop (see Figure 1).

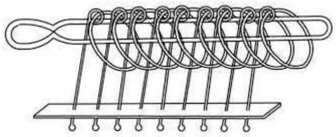

FIGURE 1: Chinese Ring puzzle

The objective of the puzzle is to remove the nine rings from the long loop. The rings are attached to the long loop so that the only possible moves are:

1. The rightmost ring can be removed or replaced on the long loop.
2. Any other ring can be removed or replaced on the loop if the ring to its right is on the loop and all the other rings to the right are off the loop.

A demonstration of the solution to the Chinese Ring puzzle can be found at [1]. Edouard Lucas discovered an elegant solution to the puzzle using binary Gray codes. A good discussion of the puzzle and its solution can be found in Knuth [2]. The 'Lichtenberg sequence' [3, A000975] gives the number of steps to solve the puzzle.

The Spinout puzzle, a variant of the Chinese Ring puzzle, was created by William Keister (see Figure 2).

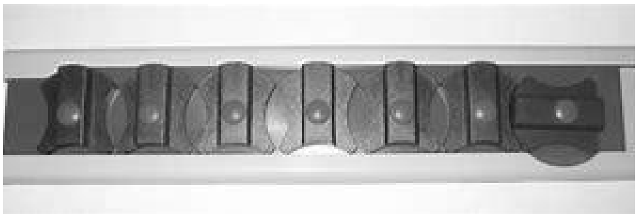

FIGURE 2: Spinout puzzle

The Spinout puzzle consists of a tray with a notch and seven spinners which can slide back and forth in the tray. The seven spinners are initially locked. The goal of the puzzle is to unlock all of the spinners, in which case the spinners can be removed from the tray. To lock or unlock a spinner, the spinner must be above the notch and the spinner immediately to its right is locked and the other spinners to its right are unlocked. The puzzle and a demonstration of some moves in the puzzle can be found at [4]. The mathematical literature contains many articles about the puzzle and its solution [5, 6].

2. *Crazy Elephant Dance puzzle*

The Crazy Elephant Dance puzzle is a generalised version of the Spinout puzzle (see Figure 3).

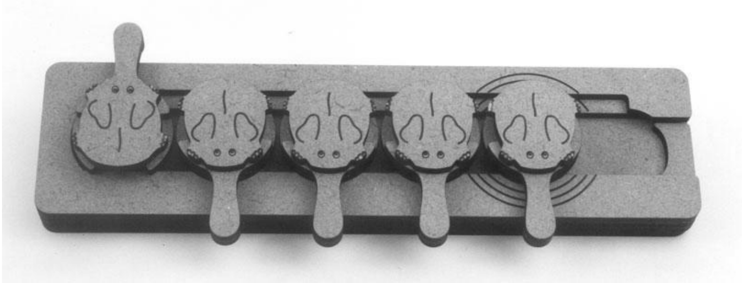

FIGURE 3: Crazy Elephant Dance puzzle

The puzzle was invented by Markus Götz and was entered in the design competition at the 25th International Puzzle Party (IPP25) held in Helsinki. A demonstration of the moves in the puzzle can be seen at [7].

The puzzle consists of a tray with a notch and five elephants in the tray. Initially, all the elephants have their snouts up. The elephants can slide in the tray but can only be removed from the tray if all the elephants have their snouts down. The goal of the puzzle is to remove all the elephants from the tray in the shortest number of moves. An elephant can only be turned to snout up, right, or down if it is located above the notch and one of the following conditions are met.

1. An elephant in the notch can turn its snout between up or right, if all the elephants to the right of the notch have their snouts down.
2. An elephant in the notch can turn its snout between right or down, if the elephant to its right has its snout up and all the other elephants to the right have their snouts down.

A demonstration of the snout moves in the Crazy Elephant Dance puzzle can be observed in [8]. A discussion of the puzzle and some of our results can be found [9].

To model the puzzle, we consider each elephant's snout as a base 3 digit; the digit 2 means the elephant's snout is up, 1 means its snout is right, and 0 means its snout is down. The general Crazy Elephant Dance puzzle consisting of $m \geqslant 1$ elephant's snouts is an m-digit base 3 number, including leading zeros. We number the digits from 1 to m, starting with the rightmost (least significant) digit. The first digit is the position of the rightmost (first) elephant's snout, the second digit is the position of the second elephant's snout, the third digit is the position of the third elephant's snout, etc. The mth digit is the position of the mth elephant's snout.

We model the movement of the snouts as an ordering of the *m*-digit base 3 numbers, starting with *m* 2's and ending with *m* 0's. The base 3 numbers in the ordering mimic the constraints in moving the snouts in the Crazy Elephant Dance puzzle. To give the constraints, let $1 \leqslant j \leqslant m$. The jth digit can change from 1 to 2 (or 2 to 1) if all the digits to its right are 0. Instead of saying 'can change from 1 to 2 (or 2 to 1)', we will say 'toggle between 1 and 2' or simply write $(1 \leftrightarrow 2)$. The jth digit can toggle between 0 and 1 if the digit to its right (the $(j-1)$th) is a 2 and all the other digits to its right are 0. It should be noted that the first digit can toggle between 0 and 1 or toggle between 1 and 2 since there are no digits to its right. We can express these values using the notation below.

Rules to change base 3 *numbers*

Let *n* be a non-negative integer. Then the endings of the base 3 numbers can change as follows.

Rule 1. $(1 \leftrightarrow 2)\underbrace{0 \ldots 0}_{n}$.

Rule 2. $(0 \leftrightarrow 1)2\underbrace{0 \ldots 0}_{n}$.

With this model, the goal of the puzzle is to produce an ordering of the base 3 *m*-digit numbers, starting with $\underbrace{2 \ldots 2}_{m}$ and using Rules 1 and 2 to find the shortest sequence to the base 3 number $\underbrace{0 \ldots 0}_{m}$.

At this point, an example of the solution to the 3 Crazy Elephant Dance puzzle may be helpful. The sequence which solves the puzzle is:

222, 221, 220, 210, 211, 212, 202, 201, 200, 100, 101, 102,

112, 111, 110, 120, 020, 010, 011, 012, 002, 001, 000.

For example, 222 can change to 221 by Rule 1 and 212 can change to 202 by Rule 2.

We will give a solution to the puzzle with *m* elephants and prove that this solution takes the minimum number of steps.

The solution changing the *m*-digit base 3 number

$$\underbrace{2 \ldots 2}_{m} \text{ to } \underbrace{0 \ldots 0}_{m}$$

according to our two rules, can be described by two recursive functions A_n and B_n. The functions are described in the following definition.

Definition 1: Let *n* be a positive integer. Let

$$A_n = \begin{cases} -1, -1 & \text{if } n = 1 \\ A_{n-1}, -n, B_{n-1}, -n, \overline{B_{n-1}} & \text{otherwise} \end{cases}$$

and

$$B_n = \begin{cases} 1,\ 1 & \text{if } n = 1 \\ B_{n-1},\ n,\ \overline{B_{n-1}},\ n & \text{otherwise.} \end{cases}$$

In this definition, n, where $n \geqslant 1$, means increase the nth digit by 1 and $-n$, where $n \geqslant 1$, means decrease the nth digit by 1. Also, $\overline{B_n}$ means reverse the steps in the B_n algorithm.

We are now ready to state our theorem.

Theorem 1: Let m be a positive integer. The function A_m changes the m-digit base 3 number from

$$\underbrace{2 \dots 2}_{m} \text{ to } \underbrace{0 \dots 0}_{m}$$

one digit at a time according to the rules of the Crazy Elephant Dance puzzle in the minimum number of steps. The function B_m changes the m digit base 3 number from

$$\underbrace{0 \dots 0}_{m} \text{ to } 2\underbrace{0 \dots 0}_{m-1}$$

one digit at a time according to the rules of the puzzle in the minimum number of steps.

With this theorem, we can solve the $m \geqslant 1$ puzzle with the function A_m.

Proof: The proof is by induction on m.

$m = 1$: Since $A_1 = -1, -1$, the number 2 becomes 1 and then 0 in two steps. This changes 2 to 0, one digit at a time according to the rules of the puzzle in the smallest number of moves. Since $B_1 = 1,\ 1$, the number 0 becomes 1 and then 2 in two steps. This changes 0 to 2, one digit at a time according to the rules of the puzzle in the smallest number of steps.

Induction Step: Assume $m > 1$ and that A_{m-1} changes

$$\underbrace{2 \dots 2}_{m-1} \text{ to } \underbrace{0 \dots 0}_{m-1},$$

one digit at a time according to the rules of the puzzle in the smallest number of steps and B_{m-1} changes

$$\underbrace{0 \dots 0}_{m-1} \text{ to } 2\underbrace{0 \dots 0}_{m-2},$$

one digit at a time and according to the rules of the puzzle in the smallest number of steps.

First, we will show that the algorithm

$$A_m = A_{m-1},\ -m,\ B_{m-1},\ -m,\ \overline{B_{m-1}}$$

changes $\underbrace{2\ldots2}_{m}$ to $\underbrace{0\ldots0}_{m}$, one digit at a time according to the rules of the puzzle in the minimum number of moves. The only way to change the most significant digit from 2 to 1 is to change the least significant $m - 1$ digits to $\underbrace{0\ldots0}_{m-1}$. Since the least significant $m - 1$ digits start at $\underbrace{2\ldots2}_{m-1}$, by the induction hypothesis, the algorithm A_{m-1} does this in the minimum number of moves. Then the move $-m$ changes the mth digit from 2 to 1. From here, the only way to change the most significant digit from 1 to 0 is to change the least significant $m - 1$ to $2\underbrace{0\ldots0}_{m-2}$. Again, by the induction hypothesis, the algorithm B_{m-1} changes the least significant $m - 1$ digits from $\underbrace{0\ldots0}_{m-1}$ to $2\underbrace{0\ldots0}_{m-2}$. And the B_{m-1} algorithm does this change in the minimum number of moves. Next, the move $-m$ changes the mth digit from 1 to 0. And finally, by the induction hypothesis, the algorithm $\overline{B_{m-1}}$ changes the least significant $m - 1$ digits from $2\underbrace{0\ldots0}_{m-2}$ to $\underbrace{0\ldots0}_{m-1}$ and in the minimum number of moves. Therefore, A_m changes $\underbrace{2\ldots2}_{m}$ to $\underbrace{0\ldots0}_{m}$, one digit at a time according to the rules of the puzzle in the minimum number of moves.

Next, we will show that the algorithm

$$B_m = B_{m-1}, m, \overline{B_{m-1}}, m$$

changes $\underbrace{0\ldots0}_{m}$ to $2\underbrace{0\ldots0}_{m-1}$, one digit at a time according to the rules of the puzzle in the minimum number of moves. The only way to change the most significant digit from 0 to 1 is to change the least significant $m - 1$ digits to $2\underbrace{0\ldots0}_{m-2}$. Since the least significant $m - 1$ digits start at $\underbrace{0\ldots0}_{m-1}$, by the induction hypothesis, the algorithm B_{m-1} does this in the minimum number of moves. Then the move m changes the mth digit from 0 to 1. From here, the only way to change the most significant digit from 1 to 2 is to change the least significant $m - 1$ digits to $\underbrace{0\ldots0}_{m-1}$. Again, by the induction hypothesis, the algorithm $\overline{B_{m-1}}$ changes the least significant $m - 1$ digits from $2\underbrace{0\ldots0}_{m-2}$ to $\underbrace{0\ldots0}_{m-1}$. And the $\overline{B_{m-1}}$ algorithm, by the induction hypothesis, does this change in the minimum number of moves. Next, the move m changes the mth digit from 1 to 2. Therefore, B_m changes $\underbrace{0\ldots0}_{m}$ to $2\underbrace{0\ldots0}_{m-1}$, one digit at a time according to the rules of the puzzle in the minimum number of moves.

This completes the proof.

Next, we give the solution to the Crazy Elephant Dance puzzle with 4 elephants. The solution is:

$$\begin{aligned}
A_4 &= A_3,\ -4,\ B_3,\ -4,\ \overline{B_3} \\
&= A_2,\ -3,\ B_2,\ -3,\ \overline{B_2},\ -4,\ B_2,\ 3,\ \overline{B_2},\ 3,\ -4,\ -3,\ B_2,\ -3,\ \overline{B_2} \\
&= A_1,\ -2,\ B_1,\ -2,\ \overline{B_1},\ -3,\ B_1,\ 2,\ \overline{B_1},\ 2,\ -3,\ -2,\ B_1,\ -2,\ \overline{B_1},\ -4,\ B_1, \\
&\quad 2,\ \overline{B_1},\ 2,\ 3,\ -2,\ B_1,\ -2,\ \overline{B_1},\ 3,\ -4,\ -3,\ B_1,\ 2,\ \overline{B_1},\ 2,\ -3, \\
&\quad -2,\ B_1,\ -2,\ \overline{B_1} \\
&= -1,\ -1,\ -2,\ 1,\ 1,\ -2,\ -1,\ -1,\ -3,\ 1,\ 1,\ 2,\ -1,\ -1,\ 2,\ -3,\ -2, \\
&\quad 1,\ 1,\ -2,\ -1,\ -1,\ -4,\ 1,\ 1,\ 2,\ -1,\ -1,\ 2,\ 3,\ -2,\ 1,\ 1,\ -2,\ -1, \\
&\quad -1,\ 3,\ -4,\ -3,\ 1,\ 1,\ 2,\ -1,\ -1,\ 2,\ -3,\ -2,\ 1,\ 1,\ -2,\ -1,\ -1.
\end{aligned}$$

The 4 digit base 3 sequence, showing each number in the solution of the 4 elephant puzzle is known as the (3, 4)-Gray code. The term Gray code comes from Frank Gray, an engineer for Bell Laboratories. A discussion of Gray codes and their history can be found in [10, 11, 12]. The (3, 4)-Gray code is given below.

2222, 2221, 2220, 2210, 2211, 2212, 2202, 2201, 2200, 2100, 2101, 2102, 2112, 2111, 2110, 2120, 2020, 2010, 2011, 2012, 2002, 2001, 2000, 1000, 1001, 1002, 1012, 1011, 1010, 1020, 1120, 1110, 1111, 1112, 1102, 1101, 1100, 1200, 0200, 0100, 0101, 0102, 0112, 0111, 0110, 0120, 0020, 0010, 0011, 0012, 0002, 0001, 0000.

The number of steps, a_n, b_n, in the sequences A_n and B_n in the Crazy Elephant Dance puzzle produces an interesting sequence. Here is numerical data for the first few Crazy Elephant Dance puzzles.

n	a_n	b_n
1	2	2
2	8	6
3	22	14
4	52	30
5	114	62
6	240	126
7	494	254
8	1004	510
9	2026	1022
10	4072	2046
11	8166	4094
12	16356	8190

These sequences satisfy the recurrence relations $a_1 = 2$, $b_1 = 2$, and for $n \geqslant 2$

$$a_n = a_{n-1} + 2b_{n-1} + 2,$$

$$b_n = 2b_{n-1} + 2.$$

The sequence a_n is [3, A005803]. The sequence b_n is [3, A000918].

We have a closed form for a_n and b_n. For $n \geqslant 1$,

$$a_n = 4 \cdot 2^n - 2n - 4,$$

$$b_n = 2(2^n - 1),$$

3. *b-spinout puzzle*

In this section, we wish to generalise the Chinese Ring puzzle and Crazy Elephant Dance puzzle to the ($b \geqslant 2$)-spinout puzzle with $m \geqslant 1$ spinners. It turns out that the Chinese Ring puzzle is the 2-spinout puzzle with 9 rings and the Crazy Elephant Dance puzzle is the 3-spinout puzzle with 5 elephants.

To model this puzzle, we consider each spinner as a base b digit; the digit $b - 1$ means the spinner is locked and the digit 0 means the spinner is unlocked. The general state of the b-spinout puzzle consisting of $m \geqslant 1$ spinners is represented as an m-digit base b number, including leading zeros. The first digit is the position of the rightmost (first) spinner, the second digit is the position of the second spinner, the third digit is the position of the third spinner, etc. The m th digit is in the position of the m th spinner.

The moves of the spinners in this puzzle can be considered as an ordering of the m-digit base b numbers, starting with m $b - 1$'s and finishing with m 0's. The base b numbers in the ordering should mimic the constraints in moving the spinners in the b-spinout puzzle. To state the constraints, let $1 \leqslant j \leqslant m$. The j th digit can toggle between $b - 2$ and $b - 1$ if all the digits to its right are 0. The j th digit can toggle between $b - 3$ and $b - 2$ if the digit to its right (the $(j - 1)$ th digit) is a $b - 1$ and all the other digits to its right are 0, etc. And, the j th digit can toggle between 0 and 1 if the digit to its right (the $(j - 1)$ th digit) is 2 and all the other digits to its right are 0. These $b - 1$ rules are listed below.

Rules to change base b numbers

Let n be a non-negative integer and $b \geqslant 2$ be an integer. Then the endings of the base b numbers can change as follows.

1. $(b - 2 \leftrightarrow b - 1)\underbrace{0 \ldots 0}_{n}.$

2. $(b - 3 \leftrightarrow b - 2)(b - 1)\underbrace{0 \ldots 0}_{n}.$

3. $(b - 4 \leftrightarrow b - 3)(b - 2)\underbrace{0 \dots 0}_{n}.$

$\vdots$

$b - 1.$ $(0 \leftrightarrow 1)\,2\underbrace{0 \dots 0}_{n}.$

With this model, the goal of the puzzle is to produce an ordering of the base b m-digit numbers, starting with $\underbrace{(b - 1) \dots (b - 1)}_{m}$ and, using the rules to change base b numbers, find the shortest sequence to the base b number $\underbrace{0 \dots 0}_{m}$.

Next, we will solve the 4-spinout puzzle. A discussion of the b-spinout puzzle for $b > 4$ is omitted. However, its solution would be similar to the solution of the b-spinout puzzles for $b \leqslant 4$.

4. *Solution to 4-spinout puzzle*

For the 4-spinout puzzle, we start with the m-digit number consisting of m 3's. This is the initial m-digit base 4 number. The following rules state how digits can change.

Rules to change base 4 *numbers*

Let n be a non-negative integer. Then the endings of the base 4 numbers can change as follows.

1. $(2 \leftrightarrow 3)\underbrace{0 \dots 0}_{n}.$
2. $(1 \leftrightarrow 2)\,3\underbrace{0 \dots 0}_{n}.$
3. $(0 \leftrightarrow 1)\,2\underbrace{0 \dots 0}_{n}.$

We note that the first digit can change from 0 to 1, or 1 to 2, or 2 to 3. The other $m - 1$ digits can only change if their conditions match one of the three conditions in the 4-spinout puzzle.

To describe the solution to the 4-spinout puzzle, we define the following three functions.

Definition 2:

Let n be a positive integer. Let

$$A_n = \begin{cases} -1, -1, -1 & \text{if } n = 1 \\ A_{n-1}, -n, B_{n-1}, -n, -(n - 1), -n, C_{n-1} & \text{otherwise,} \end{cases}$$

and

$$B_n = \begin{cases} 1,\ 1,\ 1 & \text{if } n = 1 \\ \overline{C_{n-1}},\ n,\ n-1,\ n,\ \overline{B_{n-1}},\ n & \text{otherwise}, \end{cases}$$

and

$$C_n = \begin{cases} -1,\ -1 & \text{if } n = 1 \\ B_{n-1},\ -n,\ -(n-1),\ -n,\ C_{n-1} & \text{otherwise}. \end{cases}$$

We can now state the following theorem.

Theorem 2: Let m be a positive integer. The function A_m changes the m-digit base 4 number from

$$\underbrace{3\ldots 3}_{m} \text{ to } \underbrace{0\ldots 0}_{m},$$

one digit at a time according to the rules of the 4-spinout puzzle in the minimum number of steps. The function B_m changes the m digit base 4 number from

$$\underbrace{0\ldots 0}_{m} \text{ to } 3\underbrace{0\ldots 0}_{m-1},$$

one digit at a time according to the rules of the 4-spinout puzzle in the minimum number of steps. The function C_m changes the m digit base 4 number from

$$2\underbrace{0\ldots 0}_{m-1} \text{ to } \underbrace{0\ldots 0}_{m},$$

one digit at a time according to the rules of the 4-spinout puzzle in the minimum number of steps.

The proof that these definitions solve the 4-spinout puzzle in a minimum number of moves is by induction and is similar to the proof of the Crazy Elephant Dance puzzle. We omit this proof.

We now present an example of the solution to the 4-spinout puzzle with 4 blocks. The solution is:

$$\begin{aligned} A_4 &= A_3,\ -4,\ B_3,\ -4,\ -3,\ -4,\ C_3 \\ &= A_2,\ -3,\ B_2,\ -3,\ -2,\ -3,\ C_2,\ -4,\ \overline{C_2},\ 3,\ 2,\ 3,\ \overline{B_2},\ 3,\ -4,\ -3,\ -4, \\ &\quad B_2,\ -3,\ -2,\ -3,\ C_2 \\ &= A_1,\ -2,\ B_1,\ -2,\ -1,\ -2,\ C_1,\ -3,\ \overline{C_1},\ 2,\ 1,\ 2,\ \overline{B_1},\ 2,\ -3,\ -2,\ -3, \\ &\quad B_1, -2, -1, -2, C_1, -4, \overline{C_1},\ 2,\ 1,\ 2, \overline{B_1},\ 3,\ 2,\ 3, -2, B_1, -2, -1, -2, \\ &\quad C_1,\ 3, -4, -3, -4, \overline{C_1},\ 2,\ 1,\ 2, \overline{B_1},\ 2, -3, -2, -3, B_1 - 2, -1, -2, C_1 \end{aligned}$$

$$= -1, -1 - 1 - 2,\ 1,\ 1,\ 1, -2, -1, -2, -1, -1, -3,\ 1,\ 1,\ 2,\ 1,\ 2, -1, -1,$$
$$-1,\ 2, -3, -2, -3,\ 1,\ 1,\ 1, -2, -1, -2, -1, -1, -4,\ 1,\ 1,\ 2,\ 1,\ 2, -1,$$
$$-1, -1,\ 3,\ 2,\ 3, -2,\ 1,\ 1,\ 1, -2, -1, -2, -1, -1,\ 3, -4, -3, -4,\ 1,\ 1,$$
$$2,\ 1,\ 2, -1, -1, -1,\ 2, -3, -2, -3,\ 1,\ 1,\ 1, -2, -1, -2, -1, -1.$$

The 4-digit base 4 sequence, showing each number in the solution of the 4-spinout puzzle is given below.

3333, 3332, 3331, 3330, 3320, 3321, 3322, 3323, 3313, 3312, 3302, 3301, 3300, 3200, 3201, 3202, 3212, 3213, 3223, 3222, 3221, 3220, 3230, 3130, 3120, 3020, 3021, 3022, 3023, 3013, 3012, 3002, 3001, 3000, 2000, 2001, 2002, 2012, 2013, 2023, 2022, 2021, 2020, 2120, 2130, 2230, 2220, 2221, 2222, 2223, 2213, 2212, 2202, 2201, 2200, 2300, 1300, 1200, 0200, 0201, 0202, 0212, 0213, 0223, 0222, 0221, 0220, 0230, 0130, 0120, 0020, 0021, 0022, 0023, 0013, 0012, 0002, 0001, 0000.

The number of steps, a_n, b_n, c_n, in the sequences A_n, B_n and C_n in the 4-spinout puzzle produces some interesting sequences. Here is numerical data for the first few 4-spinout puzzles.

n	a_n	b_n	c_n
1	3	3	2
2	12	9	8
3	33	21	20
4	78	45	44
5	171	93	92
6	360	189	188
7	741	381	380
8	1506	765	764
9	3039	1533	1532
10	6108	3069	3068
11	12249	6141	6140
12	24534	12285	12284

These sequences satisfy the recurrence relations $a_1 = 3$, $b_1 = 3$ and $c_1 = 2$, and for $n \geqslant 2$

$$a_n = a_{n-1} + b_{n-1} + c_{n-1} + 4,$$

$$b_n = b_{n-1} + c_{n-1} + 4,$$

$$c_n = b_{n-1} + c_{n-1} + 3.$$

The sequence a_n is a translation of [3, A000295] multiplied by 3. The sequence b_n is [3, A068156] and the sequence c_n is [3, A131128]. We have the closed forms for a_n, b_n and c_n.

For $n \geqslant 1$,

$$a_n = 6 \cdot 2^n - 3n - 6,$$
$$b_n = 3(2^n - 1),$$
$$c_n = 3(2^n - 1) - 1.$$

Acknowledgement

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CURTIS COOPER

School of CS & Math., University of Central Missouri, Warrensburg, MO 64093 USA

e-mail: *cooper@ucmo.edu*