A DICE-TOSSING PROBLEM

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1. *Introduction.*

Coin-tossing is an event often studied in courses on probability theory. One way to generalize coin-tossing problems is to consider tossing polyhedral dice with m faces labeled 0, 1, ..., m-1. The face that "scores" is the "down" face, not the "up" face which may not exist (e.g., in a tetrahedral die). Problems involving the tossing of n m-faced dice have been studied in [5] and [6]. In the first part of this paper, we will introduce a notation to represent the number of ways that a person tossing n m-faced dice can obtain a sum of k, and then develop some of the properties of this notation. The remainder of the paper will use this notation to solve a generalization to polyhedral dice of a well-known coin-tossing problem.

2. *The notation and its properties.*

Let n, m, and k be integers with $n,m \geq 1$. The symbol $\binom{n}{k}_m$ is defined as follows:

$$\binom{1}{k}_m = \begin{cases} 1, & \text{if } 0 \leq k \leq m-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\binom{n}{k}_m = \sum_{i=0}^{m-1} \binom{n-1}{k-i}_m, \qquad n > 1.$$

For $m = 2$, the quantity $\binom{n}{k}_m$ is the binomial coefficient $\binom{n}{k}$. In general, $\binom{n}{k}_m$ represents the number of ways that a person tossing n m-faced dice can obtain a sum of k. To show this, we use induction on n. The statement is clearly true for $n = 1$. Suppose it holds for some $n \geq 1$. Then the number of ways of obtaining a sum of k by tossing $n+1$ m-faced dice is the sum, for $i = 0,1,\ldots,m-1$, of the number of ways of obtaining a sum of $k-i$ by tossing n m-faced dice times the number of ways (= 1) of obtaining a score of i by tossing one m-faced die. By the induction hypothesis, this sum is equal to

$$\sum_{i=0}^{m-1} \binom{n}{k-i}_m = \binom{n+1}{k}_m,$$

and the induction is complete.

The first theorem we present will give some identities involving $\binom{n}{k}_m$.

THEOREM 1.

(i) $\binom{n}{0}_m = 1.$

(ii) $\binom{n}{1}_m = n$, if $m > 1$.

(iii) $\binom{n}{s}_m = \binom{n}{t}_m$, if $s+t = n(m-1)$.

(iv) $\sum_{k=0}^{n(m-1)} \binom{n}{k}_m = m^n$.

Proof of (i). A person tossing n m-faced dice can obtain a sum of 0 in exactly one way: all the n dice score 0.

Proof of (ii). A person tossing n m-faced dice can obtain a sum of 1 in exactly n ways: for each $i = 1,2,\ldots,n$, the ith die scores 1 and all the other dice score 0.

Proof of (iii). Let $(x_1, x_2, \ldots, x_n)$ be an ordered n-tuple where each x_i represents the score of the ith die, and suppose $x_1+x_2+\ldots+x_n = s$. Associate with this ordered n-tuple the ordered n-tuple

$$(m-1-x_1,\ m-1-x_2,\ \ldots,\ m-1-x_n).$$

Then we have a bijection from the set of all dice-tossings with a sum of s onto the set of all dice-tossings with a sum of t, where $t = n(m-1) - s$, or $s+t = n(m-1)$.

Proof of (iv). The sum on the left is the total number of outcomes of tossing n m-faced dice. Since we have n dice with m faces each, this number is m^n. □

The generating function which produces the quantity $\binom{n}{k}_m$ is

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k}_m x^k = (1 + x + \ldots + x^{m-1})^n.$$

The following theorem is proved by using this generating function.

THEOREM 2.

(i) $$\sum_{k=0}^{n(m-1)} (-1)^k \binom{n}{k}_m = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases}$$

(ii) $$\sum_{k=1}^{n(m-1)} k\binom{n}{k}_m = \frac{nm^n(m-1)}{2}.$$

(iii) $$\sum_{k=1}^{n(m-1)} (-1)^{k-1} k\binom{n}{k}_m = \begin{cases} \frac{n(1-m)}{2}, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases}$$

(iv) $$\sum_{i=0}^{k} \binom{n_1}{i}_m \cdot \binom{n_2}{k-i}_m = \binom{n_1+n_2}{k}_m.$$

Proof of (i). The sum on the left equals $f(-1)$.

Proof of (ii). The sum on the left equals $f'(1)$.

Proof of (iii). The sum on the left equals $f'(-1)$.

Proof of (iv). The result follows by equating the coefficients of x^k on both sides of the identity

$$(1+x+\ldots+x^{m-1})^{n_1}\cdot(1+x+\ldots+x^{m-1})^{n_2} = (1+x+\ldots+x^{m-1})^{n_1+n_2}. \quad \square$$

The following theorem [1] gives a method of calculating $\binom{n}{k}_m$ in terms of binomial coefficients. However, unlike [1], our proof is strictly combinatorial.

THEOREM 3. $\binom{n}{k}_m = \sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{n-1+k-mi}{n-1}$.

Proof. The quantity $\binom{n}{k}_m$ is the number of integral solutions of

$$x_1 + x_2 + \ldots + x_n = k, \tag{1}$$

where each x_i satisfies $0 \le x_i \le m-1$. Let U be the set of *all* integral solutions of (1) where each $x_i \ge 0$; and, for each $j = 1,2,\ldots,n$, let A_j be the set of integral solutions of (1) for which $x_i \ge 0$ for $1 \le i \le j-1$, $x_j \ge m$, and $x_i \ge 0$ for $j+1 \le i \le n$. With the horizontal bar denoting complementation relative to U and the vertical bars set cardinality, the principle of inclusion-exclusion [4] gives

$$\begin{aligned}\binom{n}{k}_m &= |\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n| \\ &= |U| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n| \\ &= |U| - \binom{n}{1}|A_1| + \binom{n}{2}|A_1 \cap A_2| - \binom{n}{3}|A_1 \cap A_2 \cap A_3| + \ldots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|.\end{aligned}$$

But the number of integral solutions of (1) for which $x_i \ge r_i$ for $i = 1,2,\ldots,n$ is

$$\binom{n-1+k-r_1-r_2-\ldots-r_n}{n-1}.$$

Therefore the value last obtained for $\binom{n}{k}_m$ equals

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{n-1+k-mi}{n-1},$$

as required.

3. *A dice-tossing problem with polyhedral dice.*

The principal aim of this paper is to propose and solve the following problem, which is motivated by [2], [3], and [7]:

If A tosses n $(n+m)$-faced dice and B tosses $n+m$ n-faced dice, what is the probability $P(n,m)$ that A obtains a larger sum than B?

Solution. For $0 \le r \le (n-1)(n+m)$, B can toss a sum of r in

$$\binom{n+m}{r}_n$$

ways. For each such r, A can toss a larger sum than B by tossing a sum of $r+s$ for some s with $1 \le s \le (n+m-1)n - r$, and this A can do in

$$\binom{n}{r+s}_{n+m}$$

ways. Hence the required probability is

$$P(n,m) = \sum_{r=0}^{(n-1)(n+m)} \frac{\binom{n+m}{r}_n}{n^{n+m}} \sum_{s=1}^{(n+m-1)n-r} \frac{\binom{n}{r+s}_{n+m}}{(n+m)^n}$$

$$= \frac{1}{n^{n+m}(n+m)^n} \sum_{r=0}^{(n-1)(n+m)} \sum_{s=1}^{(n+m-1)n-r} \binom{n+m}{r}_n \binom{n}{r+s}_{n+m}. \quad \square \tag{2}$$

As an example, we calculate $P(3,2)$.

$$P(3,2) = \frac{1}{3^5 \cdot 5^3} \sum_{r=0}^{10} \sum_{s=1}^{12-r} \binom{5}{r}_3 \binom{3}{r+s}_5$$

$$= \frac{1}{3^5 \cdot 5^3} (1\cdot 124 + 5\cdot 121 + 15\cdot 115 + 30\cdot 105 + 45\cdot 90 + 51\cdot 72$$

$$+ 45\cdot 53 + 30\cdot 35 + 15\cdot 20 + 5\cdot 10 + 1\cdot 4)$$

$$= \frac{17115}{30375} \approx 0.563.$$

As Klamkin notes in [3], it is doubtful that the double sum in (2) can be reduced to a "simple" single one. But for one special case the calculation is particularly easy, and the result surprising: *if* $m=1$, *then* $P(n,1) = \frac{1}{2}$ *for all* n. For

$$P(n,1) = \sum_{j>i} \frac{\binom{n}{j}_{n+1}}{(n+1)^n} \cdot \frac{\binom{n+1}{i}_n}{n^{n+1}}.$$

But the function $F(i,j) = (n^2-1-i,\ n^2-j)$ maps the set of lattice points

$$\{(i,j) \mid j > i,\ 0 \le j \le n^2,\ 0 \le i \le n^2-1\}$$

bijectively onto the set of lattice points

$$\{(i,j) \mid j \le i,\ 0 \le j \le n^2,\ 0 \le i \le n^2-1\}$$

and

$$\binom{n}{j}_{n+1} \binom{n+1}{i}_n = \binom{n}{n^2-j}_{n+1} \binom{n+1}{n^2-1-i}_n.$$

Therefore

$$P(n,1) = \frac{1}{2} \sum_{i,j} \frac{\binom{n}{j}_{n+1}}{(n+1)^n} \cdot \frac{\binom{n+1}{i}_n}{n^{n+1}} = \frac{1}{2} \cdot \frac{1}{(n+1)^n n^{n+1}} \cdot (n+1)^n n^{n+1} = \frac{1}{2}.$$

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