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DIGITAL SUM SUMS

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1. Introduction. Sums of various quantities involving arithmetic functions have been the object of much mathematical study. For example, let $d(n)$ denote the number of divisors of the positive integer n. In [1, p. 264] it is shown that

$$\sum_{n \leqslant x} d(n) = x \ln x + (2\gamma - 1)x + O(x^{\frac{1}{2}}),$$

where $\ln x$ is the natural logarithm of x and γ is Euler's constant. In [2, p, 133] we see that

$$\sum_{n \leqslant x} d(n)^2 = \frac{1}{\pi^2} x \ln^3 x + O(x \ln^2 x).$$

Also in [2, p. 133] we find that

$$\sum_{n \leqslant x} \frac{1}{(dn)} \sim A \frac{x}{\ln^{\frac{1}{2}} x}$$

for some complicated constant A.

Here we will concentrate on sums containing $s(n)$, the sum of the base 10 digits of the positive integer n. Certain sums involving $s(n)$ have already been studied. For example, in [3]

$$(1.1) \qquad \sum_{n \leqslant x} s(n) = \frac{9}{2} x \log x + O(x)$$

and in [4]

$$(1.2) \qquad \sum_{n \leqslant x} s(n)^2 = \left(\frac{9}{2}\right)^2 x \log^2 x + O(x \log x),$$

where $\log x$ denotes the base 10 logarithm of x. We will present some other digital sum sums. In particular, we will find

$$\sum_{n \leqslant x} \frac{s(n)}{[\log n] + 1}$$

and

$$\sum_{n \leqslant x} s(n)^t$$

where t is a positive integer and [.] denote the greatest integer function. To compute these sums we will make use of Abel's identity and Chebyshev's inequality. It should be noted that the results we obtain in this paper for base 10 can be extended, in a fairly straightforward way, to any base b.

2. **First Sum.** We will first show that

$$\sum_{n \leqslant x} \frac{s(n)}{[\log n] + 1} = \frac{9}{2} x + O\left(\frac{x}{\log x}\right). \tag{2.1}$$

This sum is the sum of the average digit of n as n ranges over the integers from 1 to x. Since

$$\frac{s(n)}{[\log n] + 1} = \frac{s(n)}{\log n} + \frac{s(n)}{[\log n] + 1} - \frac{s(n)}{\log n}$$

we have

$$\sum_{n \leqslant x} \frac{s(n)}{[\log n] + 1} = \sum_{n \leqslant x} \frac{s(n)}{\log n} + \sum_{n \leqslant x} \left(\frac{s(n)}{[\log n] + 1} - \frac{s(n)}{\log n}\right) \tag{2.2}$$

$$= \sum_{n \leqslant x} \frac{s(n)}{\log n} + \sum_{n \leqslant x} \frac{s(n) \cdot (\log n - [\log n] - 1)}{\log n \cdot ([\log n] + 1)}.$$

To tackle the main term of (2.2) we need a result from [5, pp. 102-103], i.e.,

$$\int_2^x \frac{dt}{\ln^k t} = O\left(\frac{x}{\ln^k x}\right). \tag{2.3}$$

for any positive integer k. Using Abel's identity [5 p. 77] and simplifying using (1.1) and (2.3) we have

$$\sum_{n \leqslant x} \frac{s(n)}{\log n} = \frac{9}{2} x + O\left(\frac{x}{\log x}\right) \tag{2.4}$$

To handle the error term of (2.2), we observe that since

$$[\log n] \leqslant \log n < [\log n] + 1,$$
$$-1 \leqslant \log n - [\log n] - 1 < 0.$$

Also since

$$s(n) \leqslant 9[\log n] + 9,$$

$$\frac{s(n)}{[\log n] + 1} \leqslant 9.$$

Putting these two ideas together, we have that

$$\sum_{n \leqslant x} \frac{s(n) \cdot (\log n - [\log n] - 1)}{\log n \cdot ([\log n] + 1)} = O\left(\sum_{n \leqslant x} \frac{1}{\log n}\right).$$

Again by Abel's identity and symplifying using (2.3) we have

$$\sum_{n \leqslant x} \frac{1}{\log n} = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Thus

(2.5) $$\sum_{n \leqslant x} \frac{s(n) \cdot (\log n - [\log x] - 1)}{\log n \cdot ([\log n] + 1)} = O\left(\frac{x}{\log x}\right).$$

Combining (2.4) and (2.5) in (2.2) we have (2.1).

3. Second Sum. We will next show that if t is a positive integer, then

(3.1) $$\sum_{n \leqslant x} s(n)^t = \left(\frac{9}{2}\right)^t x \log^t x + O(x \log^{t-\frac{1}{3}} x).$$

To begin with let

$$\mu(x) = \frac{1}{x} \sum_{n \leqslant x} s(n),$$

$$\sigma(x) = \sqrt{\frac{1}{x} \sum_{n \leqslant x} x(n)^2 - \mu(x)^2},$$

and

$$S = \{ n \leqslant x : | s(n) - \mu(x) | \leqslant k\sigma(x) \}.$$

We start symplifying the sum by writing

(3.2) $$\sum_{n \leqslant x} s(n)^t = \sum_{\substack{n \leqslant x \\ n \in S}} s(n)^t + \sum_{\substack{n \leqslant x \\ n \notin S}} s(n)^t.$$

Recall Chebyshev's inequality [6; Chapter 8], i.e.,

$$\Pr(\, | s(n) - \mu(x) | \leqslant k\sigma(x)\,) \geqslant 1 - \frac{1}{k^2}$$

for $n \leqslant x$ and any $k > 0$. Thus

$$\#S \geqslant x.\left(1 - \frac{1}{k^2}\right)$$

where $\#S$ denotes the number of elements in S. Next let $k = \log^{\frac{1}{6}} x$. Then

(3.3)
$$\#S = x + O\left(\frac{x}{\log^{\frac{1}{3}} x}\right).$$

Also by [7],

$$\mu(x) = \frac{9}{2}\log x + O(1) \qquad \text{and} \qquad \sigma(x) = O(\log^{\frac{1}{2}} x).$$

Thus for $n \in S$,

(3.4)
$$s(n) = \frac{9}{2}\log x + O(\log^{\frac{2}{3}} x)$$

and for $n \notin S$,

(3.5)
$$s(n) = O(\log x).$$

Putting (3.3), (3.4), and (3.5) in (3.2) we have

(3.6)
$$\sum_{n \leqslant x} s(n)^t = \sum_{\substack{n \leqslant x \\ n \in S}} \left(\frac{9}{2}\log x + O(\log^{\frac{2}{3}} x)\right)^t + \sum_{\substack{n \leqslant x \\ n \notin S}} O(\log)^t x)$$

$$= \#S \cdot \left(\frac{9}{2}\log x + O(\log^{\frac{2}{3}} x)\right)^t + ([x] - \#S) \cdot O(\log^t x)$$

$$= \left(x + O(\frac{x}{\log^{\frac{1}{3}} x})\right)\left(\frac{9}{2}\log x + O(\log^{\frac{2}{3}} x)\right)^t + O(x \log^{t-\frac{1}{3}} x).$$

Next using the binomial theorem and manipulating the big-oh we have that (3.6) becomes

(3.7)
$$\sum_{n \leqslant x} s(n)^t =$$

$$\left(x + O(\frac{x}{\log^{\frac{1}{3}} x})\right) \sum_{k=0}^{t} \binom{t}{k} \left(\frac{9}{2}\log x\right)^{t-k} O(\log^{\frac{2k}{3}} x) + O(x \log^{t-\frac{1}{3}} x)$$

$$= \left(x + O(\frac{x}{\log^{\frac{1}{3}} x})\right)\left(\frac{9}{2}\log x\right)^t +$$

$$\left(x + O(\frac{x}{\log^{\frac{1}{3}} x})\right) \sum_{k=1}^{t} \binom{t}{k} \left(\frac{9}{2}\log x\right)^{t-k} O(\log^{\frac{2k}{3}} x) + O(x \log^{t-\frac{1}{3}} x)$$

$$= \left(\frac{9}{2}\right)^t x \log^t x + O(x) \sum_{K=1}^{t} O(\log^{t-\frac{K}{3}} x) + O(x \log^{t-\frac{1}{3}} x)$$

$$= \left(\frac{9}{2}\right)^t x \log^t x + O(x \log^{t-\frac{1}{3}} x) + O(x \log^{t-\frac{1}{3}} x)$$

$$= \left(\frac{9}{2}\right)^t x \log^t x + O(x \log^{t-\frac{1}{3}} x).$$

This established (3.1)

4. **Questions.** We conclude the article with a conjecture and two problems. We conjecture that for any positive integer t,

$$\sum_{n \leqslant x} s(n)^t = \left(\frac{9}{2}\right)^t x \log^t x + O(x \log^{t-1} x).$$

From (1.1) and (1.2), the conjecture is true for $t=1$ and $t=2$. Another sum which may be of some interest is to find an expression for

$$\sum_{n \leqslant x} \frac{1}{s(n)}.$$

Finally, along the same lines as our first sum, we would like to evaluate

$$\sum_{n \leqslant x} \frac{\sigma(n)}{d(n)},$$

where $\sigma(n)$ is the sum of the divisors of the positive integer n.

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