DIVISIBILITY OF AN F-L TYPE CONVOLUTION

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1. Motivation

Sometimes when working on one problem, another problem and solution are found. The divisibility result in this paper is a consequence of attempts to prove some conjectures of Melham [9] related to the sum

$$L_1 L_3 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1},$$

where m is a nonnegative integer and n is a positive integer. Here, we use the usual notation for Fibonacci and Lucas numbers, i.e.

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for} \quad n \geq 2$$

and

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for} \quad n \geq 2.$$

When $m = 2$, Melham found that

$$L_1 L_3 L_5 \sum_{k=1}^{n} F_{2k}^{5} = 4F_{2n+1}^{5} - 15F_{2n+1}^{3} + 25F_{2n+1} - 14.$$

To prove this result we will use the identity

$$F_m^5 = \frac{1}{25}\left(F_{5m} - 5(-1)^m F_{3m} + 10F_m\right)$$

(proved using Binet's formula), a result by Melham [9] that if m is an odd integer

$$L_m \sum_{k=1}^{n} F_{2mk} = F_{m(2n+1)} - F_m$$

(proved using Binet's formula and summing the resulting geometric series), and the well-known identities [6]

$$F_{5n} = 25F_n^5 + 25(-1)^n F_n^3 + 5F_n \quad \text{and} \quad F_{3n} = 5F_n^3 + 3(-1)^n F_n.$$

Substituting these in turn into our sum we obtain

$$\begin{aligned}
L_1L_3L_5\sum_{k=1}^{n} F_{2k}^5 &= L_1L_3L_5\sum_{k=1}^{n}\frac{1}{25}\left(F_{10k} - 5F_{6k} + 10F_{2k}\right)\\
&= \frac{1}{25}L_1L_3L_5\left(\sum_{k=1}^{n}F_{10k} - 5\sum_{k=1}^{n}F_{6k} + 10\sum_{k=1}^{n}F_{2k}\right)\\
&= \frac{1}{25}\big(L_1L_3(F_{10n+5} - F_5) - 5L_1L_5(F_{6n+3} - F_3) + 10L_3L_5(F_{2n+1} - F_1)\big)\\
&= \frac{1}{25}\big(L_1L_3F_{10n+5} - L_1L_3F_5 - 5L_1L_5F_{6n+3} + 5L_1F_3L_5\\
&\quad + 10L_3L_5F_{2n+1} - 10F_1L_3L_5\big)\\
&= \frac{1}{25}\big(L_1L_3(25F_{2n+1}^5 - 25F_{2n+1}^3 + 5F_{2n+1}) - L_1L_3F_5\\
&\quad - 5L_1L_5(5F_{2n+1}^3 - 3F_{2n+1}) + 5L_1F_3L_5 + 10L_3L_5(F_{2n+1}) - 10F_1L_3L_5\big)\\
&= (L_1L_3)F_{2n+1}^5 - (L_1L_3 + L_1L_5)F_{2n+1}^3\\
&\quad + \frac{L_1L_3 + 3L_1L_5 + 2L_3L_5}{5}F_{2n+1} - \frac{L_1L_3F_5 - 5L_1F_3L_5 + 10F_1L_3L_5}{25}\\
&= 4F_{2n+1}^5 - 15F_{2n+1}^3 + 25F_{2n+1} - 14.
\end{aligned}$$

In the last step, we note that

$$25 \big| L_1L_3F_5 - 5L_1F_3L_5 + 10F_1L_3L_5. \tag{1}$$

Here, | means divides. This paper will generalize (1).

2. History and Result

Divisibility of Fibonacci and Lucas numbers has been the topic of much research in the mathematical literature. Some well-known divisibility properties of Fibonacci numbers and Lucas numbers can be found in [3]. For example,

$$F_n | F_m \text{ if and only if } m = kn;$$

$$L_n | F_m \text{ if and only if } m = 2kn, \quad n > 1;$$

$$\text{and} \quad L_n | L_m \text{ if and only if } m = (2k-1)n, \quad n > 1.$$

In [8], Matijasevič proved that

$$F_m^2 \big| F_{mr} \text{ if and only if } F_m \big| r.$$

Later, Hoggatt and Bicknell-Johnson [5] extended these results. In [4], Hoggatt and Bergum discovered a number of interesting results. For example, they proved that

$$n = 2 \cdot 3^k \text{ and } k \geq 1 \text{ implies } n | L_n.$$

They also showed that

$$p \text{ is an odd prime and } p | F_n \text{ implies } p^k | F_{np^{k-1}} \text{ for all } k \geq 1.$$

A corollary to this last result is the fact that

$$5^k \big| F_{5^k} \text{ for } k \geq 1.$$

In this paper we will prove the following theorem.

Theorem. Let n be a nonnegative integer. Then

$$5^n \Bigg| L_1 L_3 \cdots L_{2n+1} \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} \frac{F_{2i+1}}{L_{2i+1}}. \tag{2}$$

3. Lemmas

To prove our theorem we will need several lemmas. Some of these lemmas involve the quantity

$$a_{pj} = (-1)^j \sum_{k=j}^{p} (-1)^k 2^{p-k} \binom{p+1}{k+1} \binom{k}{j}, \tag{3}$$

where p and j are positive integers and $1 \le j \le p$. If we list the first few values of a_{pj} we have

1										
4	1									
11	5	1								
26	16	6	1							
57	42	22	7	1						
120	99	64	29	8	1					
247	219	163	93	37	9	1				
502	466	382	256	130	46	10	1			
1013	968	848	638	386	176	56	11	1		
2036	1981	1816	1486	1024	562	232	67	12	1	
4083	4017	3797	3302	2510	1586	794	299	79	13	1 .

This array is part of the sequence A008949 and can be found in [10]. Another notation we will use is $\langle\rangle$. This will denote an Eulerian number [2].

Lemma 1. Let p be a positive integer. Then

$$a_{p1} = \left\langle {p+1 \atop 1} \right\rangle.$$

Lemma 2. Let p and j be positive integers and let $1 \le j \le p$. Then

$$a_{pj} = \sum_{0 \le i \le p-j} \binom{p+1}{i}.$$

Lemma 3. Let n and k be positive integers with $n > k$. Then

$$\sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n-2i+1)^{2k+1} = 0.$$

Lemma 4. Let p and j be positive integers and $1 \le j \le p+1$. Then

$$a_{p+1,j} - \binom{p+1}{j} = 2a_{pj}.$$

Here we adopt the convention that $a_{p,p+1} = 0$.

Lemma 5. Let k and p be positive integers with $p \geq 2k$. Then

$$\sum_{j=1}^{p}(-1)^j a_{pj} j^{2k} = 0.$$

4. Proof of Lemma 1

The proof is by induction on p.

Base Step. Since

$$\begin{aligned} a_{11} &= (-1)^1 \sum_{k=1}^{1}(-1)^k 2^{1-k}\binom{2}{k+1}\binom{k}{1} \\ &= (-1)^1(-1)^1 2^{1-1}\binom{2}{2}\binom{1}{1} = 1 \end{aligned}$$

and

$$\left\langle {2 \atop 1} \right\rangle = 1,$$

the result is true for $p = 1$.

Induction Step. Assume the result is true for some positive integer p. Then by properties of binomial coefficients, the induction hypothesis, and a recurrence relation for Eulerian numbers, we have

$$\begin{aligned} a_{p+1,1} &= -\sum_{k=1}^{p+1}(-1)^k 2^{p+1-k}\binom{p+2}{k+1}\binom{k}{1} \\ &= -\sum_{k=1}^{p}(-1)^k 2^{p+1-k}\binom{p+1}{k+1}\binom{k}{1} - \sum_{k=1}^{p+1}(-1)^k 2^{p+1-k}\binom{p+1}{k}\binom{k}{1} \\ &= -2\sum_{k=1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{1} - \sum_{k=1}^{p+1}(-1)^k 2^{p+1-k}(p+1)\binom{p}{k-1} \\ &= -2\sum_{k=1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{1} + (p+1)\sum_{k=0}^{p}(-1)^k 2^{p-k}\binom{p}{k} \\ &= 2a_{p1} + (p+1)(2-1)^p = 2a_{p1} + (p+1)\cdot 1 \\ &= 2\left\langle {p+1 \atop 1} \right\rangle + (p+1)\left\langle {p+1 \atop 0} \right\rangle = \left\langle {p+2 \atop 1} \right\rangle. \end{aligned}$$

Thus, the result is true for $p+1$. By induction, the result is true for all positive integers p.

5. Proof of Lemma 2

We will prove this result in 3 parts. Let

$$c_{pj} = \sum_{0 \le i \le p-j} \binom{p+1}{i}.$$

First we will show that for any positive integer p,

$$a_{pp} = c_{pp}.$$

This follows since

$$\begin{aligned} a_{pp} &= (-1)^p \sum_{k=p}^{p} (-1)^k 2^{p-k} \binom{p+1}{k+1} \binom{k}{p} \\ &= (-1)^p (-1)^p 2^{p-p} \binom{p+1}{p+1} \binom{p}{p} = 1 \end{aligned}$$

and

$$c_{pp} = \sum_{0 \le i \le p-p} \binom{p+1}{i} = \binom{p+1}{0} = 1.$$

Second we will show that for any positive integer p,

$$a_{p1} = c_{p1}.$$

By Lemma 1

$$a_{p1} = \left\langle {p+1 \atop 1} \right\rangle.$$

By a property of Eulerian numbers

$$c_{p1} = \sum_{0 \le i \le p-1} \binom{p+1}{i} = 2^{p+1} - p - 2 = \left\langle {p+1 \atop 1} \right\rangle.$$

Third we will show that for $p \geq 2$ and $2 \leq j \leq p$,

$$a_{p+1,j} = a_{pj} + a_{p,j-1}$$

and

$$c_{p+1,j} = c_{pj} + c_{p,j-1}.$$

We see that

$$c_{p+1,j} = \sum_{0 \leq i \leq p+1-j} \binom{p+2}{i} = \sum_{0 \leq i \leq p+1-j} \binom{p+1}{i} + \sum_{1 \leq i \leq p+1-j} \binom{p+1}{i-1}$$
$$= \sum_{0 \leq i \leq p-(j-1)} \binom{p+1}{i} + \sum_{0 \leq i \leq p-j} \binom{p+1}{i} = c_{p,j-1} + c_{pj}.$$

We also see (using several binomial coefficient identities and rearranging terms in the sums) that

$$a_{p+1,j} = (-1)^j \sum_{k=j}^{p+1} (-1)^k 2^{p+1-k} \binom{p+2}{k+1} \binom{k}{j}$$
$$= 2^{p+1-j} \binom{p+2}{j+1} \binom{j}{j} + (-1)^j \sum_{k=j+1}^{p} (-1)^k 2^{p+1-k} \binom{p+2}{k+1} \binom{k}{j}$$
$$+ (-1)^j (-1)^{p+1} \binom{p+2}{p+2} \binom{p+1}{j}$$
$$= 2^{p+1-j} \binom{p+1}{j} \binom{j-1}{j-1} + 2^{p+1-j} \binom{p+1}{j+1} \binom{j}{j}$$
$$+ (-1)^j \sum_{k=j+1}^{p} (-1)^k 2^{p+1-k} \left[\binom{p+1}{k} \binom{k-1}{j} + \binom{p+1}{k} \binom{k-1}{j-1} + \binom{p+1}{k+1} \binom{k}{j} \right]$$
$$+ (-1)^j (-1)^{p+1} \binom{p+1}{p+1} \binom{p}{j-1} + (-1)^j (-1)^{p+1} \binom{p+1}{p+1} \binom{p}{j}$$
$$= 2^{p+1-j} \binom{p+1}{j} \binom{j-1}{j-1} + (-1)^j \sum_{k=j+1}^{p} (-1)^k 2^{p+1-k} \binom{p+1}{k} \binom{k-1}{j-1}$$
$$+ (-1)^j (-1)^{p+1} \binom{p+1}{p+1} \binom{p}{j-1} + 2^{p+1-j} \binom{p+1}{j+1} \binom{j}{j}$$
$$+ (-1)^j \sum_{k=j+1}^{p} (-1)^k 2^{p+1-k} \left[\binom{p+1}{k} \binom{k-1}{j} + \binom{p+1}{k+1} \binom{k}{j} \right]$$
$$+ (-1)^j (-1)^{p+1} \binom{p+1}{p+1} \binom{p}{j}$$

$$
\begin{aligned}
&= 2^{p+1-j}\binom{p+1}{j}\binom{j-1}{j-1} + (-1)^{j-1}\sum_{k=j}^{p-1}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j-1} \\
&\quad + (-1)^j(-1)^{p+1}\binom{p+1}{p+1}\binom{p}{j-1} + 2^{p+1-j}\binom{p+1}{j+1}\binom{j}{j} \\
&\quad + (-1)^j \sum_{k=j+1}^{p}(-1)^k 2^{p+1-k}\left[\binom{p+1}{k}\binom{k-1}{j} + \binom{p+1}{k+1}\binom{k}{j}\right] \\
&\quad + (-1)^j(-1)^{p+1}\binom{p+1}{p+1}\binom{p}{j} \\
&= (-1)^{j-1}\sum_{k=j-1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j-1} + 2^{p+1-j}\binom{p+1}{j+1}\binom{j}{j} - 2^{p-j}\binom{p+1}{j+1}\binom{j}{j} \\
&\quad + (-1)^j \sum_{k=j+2}^{p}(-1)^k 2^{p+1-k}\binom{p+1}{k}\binom{k-1}{j} + (-1)^j \sum_{k=j+1}^{p-1}(-1)^k 2^{p+1-k}\binom{p+1}{k+1}\binom{k}{j} \\
&\quad + (-1)^j(-1)^p 2\binom{p+1}{p+1}\binom{p}{j} + (-1)^j(-1)^{p+1}\binom{p+1}{p+1}\binom{p}{j} \\
&= (-1)^{j-1}\sum_{k=j-1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j-1} + 2^{p-j}\binom{p+1}{j+1}\binom{j}{j} \\
&\quad + (-1)^j \sum_{k=j+1}^{p-1}(-1)^{k+1} 2^{p-k}\binom{p+1}{k+1}\binom{k}{j} \\
&\quad + (-1)^j \sum_{k=j+1}^{p-1}(-1)^k 2^{p+1-k}\binom{p+1}{k+1}\binom{k}{j} + (-1)^j(-1)^p\binom{p+1}{p+1}\binom{p}{j} \\
&= (-1)^{j-1}\sum_{k=j-1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j-1} + 2^{p-j}\binom{p+1}{j+1}\binom{j}{j} \\
&\quad + (-1)^j \sum_{k=j+1}^{p-1}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j} + (-1)^j(-1)^p\binom{p+1}{p+1}\binom{p}{j} \\
&= (-1)^{j-1}\sum_{k=j-1}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j-1} + (-1)^j\sum_{k=j}^{p}(-1)^k 2^{p-k}\binom{p+1}{k+1}\binom{k}{j} \\
&= a_{p,j-1} + a_{pj}.
\end{aligned}
$$

Thus, by the 3 parts, the two arrays are identical. Therefore, the proof of Lemma 2 is complete.

6. Proof of Lemma 3

Let

$$f(i) = (2n - 2i + 1)^{2k+1}$$

and let $\triangle$ denote the forward-difference operator. Then

$$\begin{aligned}\triangle^{2n+1} f(0) &= \sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^i (2n - 2i + 1)^{2k+1} \\ &= 2 \sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n - 2i + 1)^{2k+1}.\end{aligned}$$

But since f is a polynomial in i of degree $2k + 1$ and $n > k$,

$$\triangle^{2n+1} f(0) = 0.$$

Therefore,

$$\sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n - 2i + 1)^{2k+1} = 0.$$

7. Proof of Lemma 4

Let p and j be positive integers and $1 \le j \le p + 1$. By Lemma 2

$$a_{pj} = \sum_{0 \le i \le p-j} \binom{p+1}{i}.$$

Also, assume $a_{p,p+1} = 0$. Thus,

$$\begin{aligned}a_{p+1,j} - \binom{p+1}{j} &= \sum_{0 \le i \le p+1-j} \binom{p+2}{i} - \binom{p+1}{j} \\ &= \binom{p+2}{0} + \sum_{1 \le i \le p+1-j} \binom{p+2}{i} - \binom{p+1}{j} \\ &= \binom{p+1}{0} + \sum_{1 \le i \le p+1-j} \left(\binom{p+1}{i} + \binom{p+1}{i-1} \right) - \binom{p+1}{j} \\ &= \binom{p+1}{0} + \sum_{1 \le i \le p+1-j} \binom{p+1}{i} - \binom{p+1}{p+1-j} + \sum_{1 \le i \le p+1-j} \binom{p+1}{i-1} \\ &= \binom{p+1}{0} + \sum_{1 \le i \le p-j} \binom{p+1}{i} + \sum_{0 \le i \le p-j} \binom{p+1}{i} \\ &= 2 \sum_{0 \le i \le p-j} \binom{p+1}{i} = 2a_{pj}.\end{aligned}$$

8. Proof of Lemma 5

The proof is by induction on p.

Base Step.

We will show that Lemma 5 is true for $p = 2k$. We will do this by solving a sequence of recurrence relations by the perturbation method. Let m be a nonnegative integer. Consider the recurrence relation

$$x_{-1} = 0, \quad \text{and} \quad x_n = n^m - x_{n-1} \quad \text{for} \quad n \geq 0.$$

Let $P_m(n)$ be the solution of this recurrence relation. To describe the solutions to these recurrences we need the following notation. Let $C(n)$ denote a statement which is either true or false, depending on n. Then using APL notation [2] we define

$$[C(n)] = \begin{cases} 1, & \text{if } C(n) \text{ is true} \\ 0, & \text{if } C(n) \text{ is false.} \end{cases}$$

The first 3 recurrence relations and their solutions can be found in Problem 21 of Chapter 2 of [2]. The solutions for $m = 0, 1$ and 2 are

$$\begin{aligned} P_0(n) &= 1 - [n \text{ is odd}] \\ P_1(n) &= \frac{1}{2}n + \frac{1}{2}[n \text{ is odd}] \\ \text{and} \quad P_2(n) &= \frac{1}{2}n^2 + \frac{1}{2}n. \end{aligned} \tag{4}$$

In using the perturbation method to find the solutions for $m \geq 3$, we obtain the relation

$$P_m(n) = \frac{1}{2}\left((n+1)^m - \sum_{i=1}^{m} \binom{m}{i} P_{m-i}(n)\right). \tag{5}$$

Using this relation, we can compute $P_m(n)$ for $m = 3, 4, \ldots, 12$.

$$
\begin{aligned}
P_3(n) &= \frac{1}{2}n^3 + \frac{3}{4}n^2 - \frac{1}{4}[n \text{ is odd}] \\
P_4(n) &= \frac{1}{2}n^4 + n^3 - \frac{1}{2}n \\
P_5(n) &= \frac{1}{2}n^5 + \frac{5}{4}n^4 - \frac{5}{4}n^2 + \frac{1}{2}[n \text{ is odd}] \\
P_6(n) &= \frac{1}{2}n^6 + \frac{3}{2}n^5 - \frac{5}{2}n^3 + \frac{3}{2}n \\
P_7(n) &= \frac{1}{2}n^7 + \frac{7}{4}n^6 - \frac{35}{8}n^4 + \frac{21}{4}n^2 - \frac{17}{8}[n \text{ is odd}] \\
P_8(n) &= \frac{1}{2}n^8 + 2n^7 - 7n^5 + 14n^3 - \frac{17}{2}n \\
P_9(n) &= \frac{1}{2}n^9 + \frac{9}{4}n^8 - \frac{21}{2}n^6 + \frac{63}{2}n^4 - \frac{153}{4}n^2 + \frac{31}{2}[n \text{ is odd}] \\
P_{10}(n) &= \frac{1}{2}n^{10} + \frac{5}{2}n^9 - 15n^7 + 63n^5 - \frac{255}{2}n^3 + \frac{155}{2}n \\
P_{11}(n) &= \frac{1}{2}n^{11} + \frac{11}{4}n^{10} - \frac{165}{8}n^8 + \frac{231}{2}n^6 - \frac{2805}{8}n^4 + \frac{1705}{4}n^2 - \frac{691}{4}[n \text{ is odd}] \\
P_{12}(n) &= \frac{1}{2}n^{12} + 3n^{11} - \frac{55}{2}n^9 + 198n^7 - \frac{1683}{2}n^5 + 1705n^3 - \frac{2073}{2}n.
\end{aligned}
$$

Each $P_m(n)$ is a polynomial of degree m plus possibly a term involving $[n \text{ is odd}]$. If we let b_m denote the coefficient in front of the term $[n \text{ is odd}]$ in $P_m(n)$, then we have the table of elements

m	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
b_m	-1	$1/2$	0	$-1/4$	0	$1/2$	0	$-17/8$	0	$31/2$	0	$-691/4$	0	$\cdots$

By (4) and (5), the values of the b_ms satisfy the conditions $b_0 = -1$ and for $m \geq 1$,

$$b_m = -\frac{1}{2}\sum_{i=0}^{m-1}\binom{m}{i}b_i.$$

Using generating functions, it can be shown that

$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{-2}{e^x + 1}.$$

Since

$$\frac{-2}{e^x + 1} + 1 = \frac{e^x - 1}{e^x + 1}$$

is an odd function it follows that the even subscripted bs are 0, i.e. $b_{2k} = 0$ for $k \geq 1$. Therefore, $P_{2k}(n)$ for $k \geq 1$ is a polynomial of degree $2k$, i.e. it contains no term $[n$ is odd$]$.

It should be noted that the Genocchi numbers [1] are defined by

$$\frac{2x}{e^x+1} = \sum_{k=0}^{\infty} G_k \frac{x^k}{k!}.$$

Therefore, for $n \geq 0$

$$b_n = -\frac{1}{n+1} G_{n+1}.$$

Now, using Lemma 2 on the first equality we have

$$\begin{aligned}
\sum_{j=1}^{2k} (-1)^j a_{2k,j} j^{2k} &= \sum_{j=1}^{2k} (-1)^j \sum_{i=0}^{2k-j} \binom{2k+1}{i} j^{2k} \\
&= \sum_{i=0}^{2k-1} \binom{2k+1}{i} \sum_{j=1}^{2k-i} (-1)^j j^{2k} \\
&= \sum_{i=0}^{2k-1} \binom{2k+1}{i} \sum_{j=0}^{2k-i} (-1)^j j^{2k} \\
&= \sum_{i=0}^{2k+1} \binom{2k+1}{i} \sum_{j=0}^{2k-i} (-1)^j j^{2k} \\
&= \sum_{i=0}^{2k+1} \binom{2k+1}{2k+1-i} \sum_{j=0}^{2k-(2k+1-i)} (-1)^j j^{2k} \\
&= \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{i+1} \left(\sum_{j=0}^{i-1} (-1)^j j^{2k} (-1)^{i+1} \right) \\
&= \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{i+1} P_{2k}(-1+i).
\end{aligned}$$

But since the last sum is $-\triangle^{2k+1} P_{2k}(-1)$ and P_{2k} is a polynomial of degree $2k$, it follows that the above sum is 0. This completes the proof of the base step.

Induction Step. Next, we will show that if the formula is true for some $p \geq 2k$, then it is true for $p+1$. Suppose that the formula is true for some $p \geq 2k$. We will use the fact that

$$\sum_{j=0}^{p+1} (-1)^{j+1} \binom{p+1}{j} j^{2k} = 0.$$

This can be seen by noting that if $Q(j) = j^{2k}$, then

$$\sum_{j=0}^{p+1} (-1)^{j+1} \binom{p+1}{j} j^{2k} = -\triangle^{p+1} Q(0) = 0$$

since Q is a polynomial in j of degree $2k$ and $p+1 > 2k$. Hence,

$$\begin{aligned}
&\sum_{j=1}^{p+1} (-1)^j a_{p+1,j} j^{2k} \\
&= \sum_{j=1}^{p+1} (-1)^j a_{p+1,j} j^{2k} + \sum_{j=0}^{p+1} (-1)^{j+1} \binom{p+1}{j} j^{2k} \\
&= \sum_{j=1}^{p+1} (-1)^j a_{p+1,j} j^{2k} + \sum_{j=1}^{p+1} (-1)^{j+1} \binom{p+1}{j} j^{2k} \\
&= \sum_{j=1}^{p+1} (-1)^j \left(a_{p+1,j} - \binom{p+1}{j} \right) j^{2k} \\
&= \sum_{j=1}^{p} (-1)^j 2 a_{pj} j^{2k} = 2 \left(\sum_{j=1}^{p} (-1)^j a_{pj} j^{2k} \right).
\end{aligned}$$

The next to last equality follows from Lemma 4. But the last expression is 0 by our induction hypothesis. Therefore, the result is true for $p+1$. This completes the proof of the induction step.

Thus, by induction, Lemma 5 is proved.

9. Proof of the Theorem

We begin the proof of (2) by noting that if

$$(x-1)^{2n+1} \Big| (x+1)(x^3+1)\cdots(x^{2n+1}+1) \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} \frac{x^{2i+1}-1}{x^{2i+1}+1} \qquad (6)$$

is true, then (2) is true. Suppose (6) is true and substitute α/β for x in (6), where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Using the fact that $\alpha - \beta = \sqrt{5}$ and multiplying (6) by β^{n^2}, (6) becomes

$$5^n \Big| (\alpha+\beta)(\alpha^3+\beta^3)\cdots(\alpha^{2n+1}+\beta^{2n+1}) \sum_{i=0}^{n} \binom{2n+1}{n-i}(-1)^{n-i} \frac{\alpha^{2i+1}-\beta^{2i+1}}{\sqrt{5}(\alpha^{2i+1}+\beta^{2i+1})}.$$

But this last result, by the use of Binet's formula [3], i.e.

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

is (2) .

Let

$$f(x) = (x+1)(x^3+1)\cdots(x^{2n+1}+1)$$

and

$$g(x) = \sum_{i=0}^{n} \binom{2n+1}{n-i}(-1)^{n-i} \frac{x^{2i+1}-1}{x^{2i+1}+1}.$$

Now, if D denotes the derivative operator, then by applying the product rule j times we obtain the formula

$$D^j f(x)g(x) = \sum_{i=0}^{j} \binom{j}{i} D^i f(x) D^{j-i} g(x). \tag{7}$$

Proving (6) would be equivalent to showing that

$$D^j f(1)g(1) = 0 \text{ for } j = 0, 1, \ldots, 2n. \tag{8}$$

But by (7) we can prove (8) if we can show that

$$g(1) = Dg(1) = D^2 g(1) = \cdots = D^{2n} g(1) = 0. \tag{9}$$

Simplifying $g(x)$ we have

$$\begin{aligned} g(x) &= \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} \frac{x^{2i+1}-1}{x^{2i+1}+1} \\ &= \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} \left(1 - \frac{2}{x^{2i+1}+1}\right). \end{aligned} \tag{10}$$

First of all, it is clear that $g(1) = 0$. To compute the pth derivative of $g(x)$ where $1 \le p \le 2n$, we need to find the pth derivative of

$$\frac{1}{x^{2i+1}+1}.$$

Using a result in [7],

$$D^p \left[\frac{1}{x^{2i+1}+1}\right] = \sum_{k=1}^{p} (-1)^k \binom{p+1}{k+1} \frac{1}{(x^{2i+1}+1)^{k+1}} D^p \left[(x^{2i+1}+1)^k\right].$$

We now need the notation for falling factorials [2], i.e.

$$x^{\underline{p}} = x(x-1)\cdots(x-p+1)$$

and the binomial theorem

$$(x^{2i+1}+1)^k = \sum_{j=0}^{k} \binom{k}{j} x^{(2i+1)j}.$$

Thus,

$$\begin{aligned} D^p \left[\sum_{j=0}^{k} \binom{k}{j} x^{(2i+1)j}\right] &= \sum_{j=0}^{k} \binom{k}{j} D^p x^{(2i+1)j} \\ &= \sum_{j=0}^{k} \binom{k}{j} [(2i+1)j][(2i+1)j-1]\cdots[(2i+1)j-p+1] x^{(2i+1)j-p} \\ &= \sum_{j=0}^{k} \binom{k}{j} [(2i+1)j]^{\underline{p}} x^{(2i+1)j-p}. \end{aligned}$$

It follows that

$$D^p\left[\frac{1}{x^{2i+1}+1}\right]\Bigg|_{x=1}=\sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}2^{-k-1}\sum_{j=0}^{k}\binom{k}{j}[(2i+1)j]^{\underline{p}}. \qquad (11)$$

Next, we will study (11) with $2i+1$ replaced by m, i.e.

$$\sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}2^{-k-1}\sum_{j=0}^{k}\binom{k}{j}(jm)^{\underline{p}}.$$

Using the fact that $p \geq 1$, so we have no term when $j=0$, we wish to investigate the sum

$$\sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}2^{-k-1}\sum_{j=1}^{k}\binom{k}{j}(jm)^{\underline{p}}. \qquad (12)$$

By changing the order of summation, it follows that (12) becomes

$$\begin{aligned}&\sum_{j=1}^{p}(jm)^{\underline{p}}\sum_{k=j}^{p}(-1)^k\binom{p+1}{k+1}\binom{k}{j}2^{-k-1}\\&=\frac{1}{2^{p+1}}\sum_{j=1}^{p}(jm)^{\underline{p}}\sum_{k=j}^{p}(-1)^k2^{p-k}\binom{p+1}{k+1}\binom{k}{j}.\end{aligned}$$

We want to show that the above polynomial in m only contains odd terms, i.e. there are only terms of odd degree in the polynomial. The first few such polynomials are

$$\begin{aligned}&\frac{1}{4}(-m),\\&\frac{1}{8}(2m),\\&\frac{1}{16}(2m^3-8m),\\&\frac{1}{32}(-24m^3+48m),\\&\frac{1}{64}(-16m^5+280m^3-384m),\\\text{and}\quad&\frac{1}{128}(480m^5-3600m^3+3840m),\end{aligned}$$

for $p = 1, 2, 3, 4, 5$, and 6, respectively. Now, by (3) we have that the polynomial is

$$D^p\left[\frac{1}{x^m+1}\right]\Bigg|_{x=1}=\frac{1}{2^{p+1}}\sum_{j=1}^{p}(-1)^j a_{pj}(jm)^{\underline{p}}.$$

Next, we recall the Stirling numbers of the first kind. They are denoted by

$$s(n,k)$$

and count the number of ways to arrange n objects into k cycles [1,2]. A property of Stirling numbers of the first kind is

$$s(n,n-k) = \sum_{0 \le i_1 < \cdots < i_k \le n-1} i_1 \cdots i_k.$$

Thus, we have that

$$x^{\underline{p}} = x(x-1)\cdots(x-p+1) = \sum_{j=0}^{p} (-1)^j s(p,p-j) x^{p-j}.$$

It follows that

$$(jm)^{\underline{p}} = \sum_{k=0}^{p} (-1)^k = s(p,p-k)(jm)^{p-k} = \sum_{k=0}^{p} (-1)^k s(p,p-k) j^{p-k} m^{p-k}. \quad (13)$$

Hence, by using (13) and changing the order of summation, the polynomial in m is

$$\begin{aligned}
& D^p \left[\frac{1}{x^m+1} \right] \Bigg|_{x=1} \\
&= \frac{1}{2^{p+1}} \sum_{j=1}^{p} (-1)^j a_{pj} (jm)^{\underline{p}} \\
&= \frac{1}{2^{p+1}} \sum_{j=1}^{p} (-1)^j a_{pj} \sum_{k=0}^{p} (-1)^k s(p,p-k) j^{p-k} m^{p-k} \\
&= \frac{1}{2^{p+1}} \sum_{k=0}^{p} (-1)^k s(p,p-k) m^{p-k} \sum_{j=1}^{p} (-1)^j a_{pj} j^{p-k} \\
&= \frac{1}{2^{p+1}} \sum_{k=0}^{p} (-1)^{p-k} s(p,k) m^k \sum_{j=1}^{p} (-1)^j a_{pj} j^k.
\end{aligned}$$

Therefore, for $p \geq 1$ we have by (7) that

$$
\begin{aligned}
D^p g(1) &= \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} D^p \left(1 - \frac{2}{g_{2i+1}(x)}\right)\Bigg|_{x=1} \\
&= \sum_{i=0}^{n} \binom{2n+1}{i} (-1)^{i} D^p \left(1 - \frac{2}{g_{2n-2i+1}(x)}\right)\Bigg|_{x=1} \\
&= -2\sum_{i=0}^{n} \binom{2n+1}{i} (-1)^{i} D^p \left(\frac{1}{g_{2n-2i+1}(x)}\right)\Bigg|_{x=1} \\
&= -2\sum_{i=0}^{n} \binom{2n+1}{i} (-1)^{i} \frac{1}{2^{p+1}} \sum_{k=0}^{p} (-1)^{p-k} s(p,k)(2n-2i+1)^k \sum_{j=1}^{p} (-1)^j a_{pj} j^k \\
&= \frac{-2}{2^{p+1}} \sum_{k=0}^{p} (-1)^{p-k} s(p,k) \sum_{j=1}^{p} (-1)^j a_{pj} j^k \sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n-2i+1)^k.
\end{aligned}
$$

To finish the proof of the Theorem we will prove that the last expression is 0. To do this we will isolate the term when $k = 0$ and the two sums when $0 < 2k+1 \leq p$ and $0 < 2k \leq p$. The term and the two sums are listed below.

$$
\begin{aligned}
&\frac{-2}{2^{p+1}} (-1)^p s(p,0) \sum_{j=1}^{p} (-1)^j a_{pj} \sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i \\
&+ \frac{-2}{2^{p+1}} \sum_{0<2k+1\leq p} (-1)^{p-2k-1} s(p,2k+1) \sum_{j=1}^{p} (-1)^j a_{pj} j^{2k+1} \\
&\quad \left(\sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n-2i+1)^{2k+1} \right) \\
&+ \frac{-2}{2^{p+1}} \sum_{0<2k\leq p} (-1)^{p-2k} s(p,2k) \left(\sum_{j=1}^{p} (-1)^j a_{pj} j^{2k} \right) \\
&\quad \sum_{i=0}^{n} \binom{2n+1}{i} (-1)^i (2n-2i+1)^{2k}.
\end{aligned}
$$

The term when $k = 0$ is 0 since $s(p,0) = 0$ for $p \geq 1$. Since $1 \leq p \leq 2n$ and $2k+1 \leq p$, it follows that $k < n$. Thus by Lemma 3 the first sum is 0. Lemma 5 proves that the second sum is 0.

Summarizing, we have just shown that the term and the two sums are 0. Thus, for $1 \le p \le 2n$ we have $D^p g(1) = 0$. Since $g(1) = 0$ we have proved that (6) is true. Therefore, the Theorem is proved.

10. Further Questions

First of all, we could study the polynomial P_m in Lemma 5. Is there an explicit formula for P_m? Second, in studying (2) we came across the conjecture that

$$(x+1)^n \Big| (x+1)(x^3+1)\cdots(x^{2n+1}+1) \sum_{i=0}^{n} \binom{2n+1}{n-i} (-1)^{n-i} \frac{x^{2i+1}-1}{x^{2i+1}+1}.$$

Finally, we could again study Melham's sum

$$L_1 L_3 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1},$$

where m is a nonnegative integer and n is a positive integer.

References

1. L. Comtet. *Advanced Combinatorics.* Dordrecht: D. Reidel, 1974.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics.* Reading, Mass.: Addison-Wesley, 1994.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969.
4. V. E. Hoggatt, Jr. & G. E. Bergum. "Divisibility and Congruence Relations." *The Fibonacci Quarterly* **12.2** (1974): 189–195.
5. V. E. Hoggatt, Jr. & Marjorie Bicknell-Johnson. "Divisibility by Fibonacci and Lucas Squares." *The Fibonacci Quarterly* **15.1** (1977): 3–8.
6. D. Jennings. "Some Polynomial Identities for the Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **31.2** (1993): 134–137.
7. R. A. Leslie. "How Not to Repeatedly Differentiate a Reciprocal." *American Mathematical Monthly* **98** (1991): 732–735.
8. Y. V. Matijasevič. "Enumerable Sets are Diophantine." *Proc. of the Academy of Sciences of the USSR* **11** (1970): No. 2.

9. R. S. Melham. private communication.

10. N. J. A. Sloane. *On-Line Encyclopedia of Integer Sequences.* Published electronically at http://www.research.att.com/∼njas/sequences.

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