VARIATIONS ON A FIVE-BY-FIVE SEATING REARRANGEMENT PROBLEM

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1. Introduction. In [1], the following problem was presented.

> A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). Determine whether or not this directive can be carried out.

There is no way that this directive can be carried out. To see this, consider the desk grid as a 5×5 checkerboard of X and O squares with an X in the upper left corner.

X	O	X	O	X
O	X	O	X	O
X	O	X	O	X
O	X	O	X	O
X	O	X	O	X

Now suppose we could carry out this directive. Then the pupils in the X squares must move to the O squares. But, since there are more X squares than O squares, the pigeonhole principle [2] says that two pupils in X squares must move to the same O square, a contradiction. In fact, there is no way to carry out this directive if the classroom has m rows of n desks per row and both m and n are odd.

Here, we are interested in the following variation to the problem.

> A classroom has m rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). *How many ways* can this directive be carried out?

First, we will show that the answer to the $2 \times n$ problem is a surprising F_{n+1}^2, where F_n denotes the nth Fibonacci number. Next, we will study the problem with 3 rows and $2n$ desks per row. Finally, we will ask some related questions.

2. Terminology. To begin studying the $2 \times n$ problem, we define the term "$2 \times n$ seating rearrangement." For example, we would like

$$\begin{array}{ccccccccccc}
\bigcirc & & \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc & \leftarrow & \bigcirc & & \bigcirc & \rightarrow & \bigcirc \\
\updownarrow & & & & & & \downarrow & & \uparrow & & \uparrow & & \downarrow \\
\bigcirc & & \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc & \rightarrow & \bigcirc & & \bigcirc & \leftarrow & \bigcirc
\end{array}$$

to be a 2×7 seating rearrangement. In this notation, $\bigcirc$ denotes a desk and $\rightarrow$ denotes the movement of a pupil from one desk to another.

Let n be a positive integer. Suppose there are $2n$ desks arranged in 2 rows and n columns and that pupils are seated at each desk in this array of desks. A $\underline{2 \times n \text{ seating rearrangement}}$ is any movement of pupils among the $2n$ desks so that each desk is vacated by one pupil and reoccupied by a different pupil and each pupil moves to a horizontal or vertical neighboring desk.

Thus,

$$\begin{array}{c} \bigcirc \\ \updownarrow \\ \bigcirc \end{array}$$

$$\begin{array}{ccc} \bigcirc & \rightarrow & \bigcirc \\ \uparrow & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc \end{array}$$

$$\begin{array}{ccccc} \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ \downarrow & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \end{array}$$

$$\begin{array}{ccccccccc} \bigcirc & \rightarrow & \bigcirc & & \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ \uparrow & & \downarrow & & \updownarrow & & & & \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array}$$

are examples of 2×1, 2×2, 2×3, and 2×5 seating rearrangements, respectively.

3. Solution to the $2 \times n$ Problem. The set T_n of all $2 \times n$ rearrangements may be partitioned into four subsets A_n, B_n, C_n, D_n according to the four ways in which they end as follows:

$$A_n : \begin{array}{ccc} \bigcirc & \cdots & \bigcirc \\ & & \updownarrow \\ \bigcirc & \cdots & \bigcirc \end{array}$$

$$B_n : \begin{array}{ccccc} \bigcirc & \cdots & \bigcirc & \rightarrow & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \cdots & \bigcirc & \leftarrow & \bigcirc \end{array}$$

$$C_n : \begin{array}{ccccc} \bigcirc & \cdots & \bigcirc & \leftarrow & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \cdots & \bigcirc & \rightarrow & \bigcirc \end{array}$$

$$D_n : \begin{array}{l} \bigcirc \cdots \bigcirc \leftrightarrow \bigcirc \\ \\ \bigcirc \cdots \bigcirc \leftrightarrow \bigcirc \end{array}$$

Let the cardinalities of A_n, B_n, C_n, D_n, T_n, respectively, be a_n, b_n, c_n, d_n, and t_n; the number we wish to determine, then, is

$$t_n = a_n + b_n + c_n + d_n.$$

Now, these numbers are governed by the recursions

$$\begin{array}{rl} (i) & a_{n+1} = t_n = a_n + b_n + c_n + d_n \\ (ii) & b_{n+1} = a_n + b_n \\ (iii) & c_{n+1} = a_n + c_n \\ (iv) & d_{n+1} = a_n. \end{array}$$

To see (i) we need only observe that any member of T_n becomes a member of A_{n+1} when the extra column

$$\begin{array}{c} \bigcirc \\ \updownarrow \\ \bigcirc \end{array}$$

is attached to the end, and conversely. Recursion (ii) is established by the following 1-1 correspondences between $A_n \cup B_n$ and B_{n+1}:

$$\begin{array}{ccc} \begin{array}{r} \bigcirc \cdots \bigcirc \\ \updownarrow \\ \bigcirc \cdots \bigcirc \end{array} \in A_n \leftrightarrow & \begin{array}{ccccc} \bigcirc \cdots & \bigcirc & \rightarrow & \bigcirc \\ & \uparrow & & \downarrow \\ \bigcirc \cdots & \bigcirc & \leftarrow & \bigcirc \end{array} \end{array}$$

$$\begin{array}{ccc} \begin{array}{r} \bigcirc \cdots \bigcirc \rightarrow \bigcirc \\ \downarrow \\ \bigcirc \cdots \bigcirc \leftarrow \bigcirc \end{array} \in B_n \leftrightarrow & \begin{array}{r} \bigcirc \cdots \bigcirc \rightarrow \bigcirc \rightarrow \bigcirc \\ \downarrow \\ \bigcirc \cdots \bigcirc \leftarrow \bigcirc \leftarrow \bigcirc \end{array} \end{array}$$

Similar correspondences establish (iii), and (iv) is given immediately by

$$\begin{array}{r} \bigcirc\cdots\bigcirc \\ \updownarrow \\ \bigcirc\cdots\bigcirc \end{array} \in A_n \leftrightarrow \begin{array}{c} \bigcirc\cdots\bigcirc\leftrightarrow\bigcirc \\ \\ \bigcirc\cdots\bigcirc\leftrightarrow\bigcirc \end{array} \in D_{n+1}.$$

These results are summarized nicely in the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix}.$$

With

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

we have $t_2 = 4$, and then

$$\begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix},$$

making $t_3 = 4 + 2 + 2 + 1 = 9$. In fact, an easy induction yields the general result

$$\begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} F_n^2 \\ F_{n-1}F_n \\ F_{n-1}F_n \\ F_{n-1}^2 \end{pmatrix},$$

making

$$t_n = a_{n+1} = a_n + b_n + c_n + d_n = F_{n+1}^2,$$

where F_n is the nth Fibonacci number.

4. The $3 \times 2n$ Problem. Next we consider the $3 \times 2n$ problem.

A classroom has 3 rows of $2n$ desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). How many ways can this directive can be carried out?

The set Z_{2n} of all $3 \times 2n$ rearrangements may be partitioned into twenty-five subsets A_{2n}, B_{2n}, C_{2n}, $\cdots$, Y_{2n} according to the twenty-five ways in which the last of the $2n$ columns end as follows:

$$A_{2n}: \begin{array}{ccc} \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc \end{array} \qquad B_{2n}: \begin{array}{ccc} \bigcirc & \leftarrow & \bigcirc \\ \downarrow & & \uparrow \\ \bigcirc & & \bigcirc \\ \downarrow & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc \end{array} \qquad C_{2n}: \begin{array}{ccc} \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & & \bigcirc \\ \updownarrow & & \updownarrow \\ \bigcirc & & \bigcirc \end{array}$$

$$D_{2n}: \begin{array}{ccc} \bigcirc & & \bigcirc \\ \updownarrow & & \updownarrow \\ \bigcirc & & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc \end{array} \qquad E_{2n}: \begin{array}{ccc} \bigcirc & \rightarrow & \bigcirc \\ \uparrow & & \downarrow \\ \bigcirc & & \bigcirc \\ \uparrow & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc \end{array} \qquad F_{2n}: \begin{array}{ccc} \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \leftarrow & \bigcirc \\ \downarrow & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc \end{array}$$

$$G_{2n}: \begin{array}{ccc} \bigcirc & \leftarrow & \bigcirc \\ \downarrow & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc \end{array} \qquad H_{2n}: \begin{array}{ccc} \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \rightarrow & \bigcirc \\ \uparrow & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc \end{array} \qquad I_{2n}: \begin{array}{ccc} \bigcirc & \rightarrow & \bigcirc \\ \uparrow & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc \end{array}$$

$$J_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftarrow & \bigcirc \\ & & \downarrow & & \uparrow \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \end{array} \qquad K_{2n}: \begin{array}{ccccc} \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & & \bigcirc \\ & & \downarrow & & \uparrow \\ \bigcirc & & \bigcirc & \rightarrow & \bigcirc \end{array} \qquad L_{2n}: \begin{array}{ccccc} \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \end{array}$$

$$M_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc \\ & & \uparrow & & \updownarrow \\ \bigcirc & \rightarrow & \bigcirc & & \bigcirc \end{array} \qquad N_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \rightarrow & \bigcirc & & \bigcirc \\ & & \downarrow & & \updownarrow \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc \end{array} \qquad O_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \\ & & & & \updownarrow \\ \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \end{array}$$

$$P_{2n}: \begin{array}{ccccc} \bigcirc & \leftarrow & \bigcirc & & \bigcirc \\ & & \uparrow & & \updownarrow \\ \bigcirc & \rightarrow & \bigcirc & & \bigcirc \\ & & & & \\ \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array} \quad Q_{2n}: \begin{array}{ccccc} \bigcirc & \rightarrow & \bigcirc & & \bigcirc \\ & & \downarrow & & \updownarrow \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc \\ & & & & \\ \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array} \quad R_{2n}: \begin{array}{ccccc} \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \\ & & & & \updownarrow \\ \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \\ & & & & \\ \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array}$$

$$S_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftarrow & \bigcirc \\ & & \uparrow & & \downarrow \\ \bigcirc & \rightarrow & \bigcirc & & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \end{array} \quad T_{2n}: \begin{array}{ccccc} \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc & & \bigcirc \\ & & \uparrow & & \downarrow \\ \bigcirc & & \bigcirc & \leftarrow & \bigcirc \end{array} \quad U_{2n}: \begin{array}{ccccc} \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftrightarrow & \bigcirc & & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \end{array}$$

$$V_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ & & & & \\ \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \end{array} \quad W_{2n}: \begin{array}{ccccc} \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ & & & & \uparrow \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \\ & & & & \\ \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array}$$

$$X_{2n}: \begin{array}{ccccc} \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \\ & & & & \\ \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \end{array} \quad Y_{2n}: \begin{array}{ccccc} \bigcirc & \rightarrow & \bigcirc & \rightarrow & \bigcirc \\ & & & & \downarrow \\ \bigcirc & \leftarrow & \bigcirc & \leftarrow & \bigcirc \\ & & & & \\ \bigcirc & & \bigcirc & \leftrightarrow & \bigcirc \end{array}$$

Let the cardinalities of A_{2n}, B_{2n}, C_{2n}, $\cdots$, Y_{2n}, and Z_{2n} respectively, be a_{2n}, b_{2n}, c_{2n}, $\cdots$, y_{2n} and z_{2n}; the number we wish to determine, then, is

$$z_{2n} = a_{2n} + b_{2n} + c_{2n} + \cdots + y_{2n}.$$

Next, by examining the 1-1 correspondences between the different $3 \times 2(n+1)$ seating rearrangements and the $3 \times 2n$ seating rearrangements, we obtain, A, the

25×25 matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The proof that

$$
\begin{pmatrix} a_{2(n+1)} \\ b_{2(n+1)} \\ c_{2(n+1)} \\ \vdots \\ y_{2(n+1)} \end{pmatrix} = A \begin{pmatrix} a_{2n} \\ b_{2n} \\ c_{2n} \\ \vdots \\ y_{2n} \end{pmatrix}
$$

is very long and tedious, but is similar to the proof of the $2 \times n$ seating rearrangement problem's proof. The initial conditions for the $3 \times 2n$ problem are

$$a_2 = b_2 = c_2 = \cdots = i_2 = 1$$

and

$$j_2 = k_2 = \cdots = y_2 = 0.$$

5. Solution to the $3 \times 2n$ Problem. It can be shown, by induction on n, that

$$a_{2n} = b_{2n} = c_{2n} = \cdots = i_{2n}$$

$$j_{2n} = k_{2n} = m_{2n} = n_{2n} = p_{2n} = q_{2n} = s_{2n} = t_{2n} = v_{2n} = \cdots = y_{2n}$$

$$l_{2n} = o_{2n} = r_{2n} = u_{2n}.$$

Therefore, it follows that

$$\begin{pmatrix} a_{2(n+1)} \\ j_{2(n+1)} \\ l_{2(n+1)} \end{pmatrix} = \begin{pmatrix} 9 & 12 & 4 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_{2n} \\ j_{2n} \\ l_{2n} \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^3 - 15\lambda^2 + 15\lambda - 1.$$

The zeros of this polynomial are 1, $7 + 4\sqrt{3}$, and $7 - 4\sqrt{3}$. Assuming

$$z_{2n} = A + B(7 + 4\sqrt{3})^n + C(7 - 4\sqrt{3})^n$$

and using the initial conditions, $z_2 = 9$, $z_4 = 121$, $z_6 = 1681$, we have that

$$A = \frac{1}{3}, \quad B = \frac{2 + \sqrt{3}}{6}, \quad \text{and} \quad C = \frac{2 - \sqrt{3}}{6}.$$

With this guess it can be shown, again by induction on n, that

$$z_{2n} = \frac{2 + \sqrt{3}}{6}(7 + 4\sqrt{3})^n + \frac{2 - \sqrt{3}}{6}(7 - 4\sqrt{3})^n + \frac{1}{3}.$$

6. Questions. Finally, we state some open questions.

A classroom has 2 rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front or behind, the one to the left or the right, or to one of the diagonal seats (of course, not all these options are possible for all students). *How many ways* can this directive be carried out?

A classroom has a triangular array of $\frac{n(n+1)}{2}$ desks. The teacher requests each pupil to change his seat by going the closest seat in any direction. *How many ways* can this directive be carried out?

And finally, we still have open the original question which started it all.

A classroom has m rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). *How many ways* can this directive be carried out?

References

1. R. Honsberger, *Mathematical Gems I*, The Dolciani Mathematical Expositions, No. 1, The Mathematical Association of America, p. 76.
2. C. L. Liu, *Elements of Discrete Mathematics*, McGraw-Hill Book Company, New York, 1985, p. 127.
3. P. Henrici, *Elements of Numerical Analysis*, John Wiley & Sons, Inc., New York, 1964, p. 121.