AN IDENTITY FOR GENERALIZED FIBONACCI NUMBERS

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Abstract. We will present an identity for the generalized Fibonacci numbers.

1. Introduction

Fibonacci and generalized Fibonacci identities have been studied by many mathematicians for many years. For example, the Gelin-Cesàro identity [1] states that if F_n denotes the nth Fibonacci number, $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, then for any integer $n \geq 2$,

$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1.$$

Let a, b, p, and q be real numbers. A generalized Fibonacci sequence is defined as W_n, where $W_0 = a$, $W_1 = b$, and for $n \geq 2$, $W_n = pW_{n-1} - qW_{n-2}$. Then Melham [3] showed that for $n \geq 0$,

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1}),$$

where $e = pab - qa^2 - b^2$.

In [2], Howard defined a generalized Tribonacci sequence for $n \geq 3$ as

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},$$

where V_0, V_1, and V_2 are arbitrary complex numbers, and r, s, and t are arbitrary integers, with $t \neq 0$. The sequence can be extended in the usual way to negative subscripts. Using the notation

$$V_n = V_n(V_0, V_1, V_2; r, s, t)$$

to emphasis the initial conditions and coefficients, he also defined

$$J_n = V_n(3, r, r^2 + 2s; r, s, t).$$

Howard then proceeded to prove the identity

$$V_{n+2m} = J_m V_{n+m} - t^m J_{-m} V_n + t^m V_{n-m},$$

where n and m are arbitrary integers.

In this paper we state and prove an identity for the k-generalized Fibonacci numbers.

Definition 1.1. *Let $r \geq 2$ be an integer. The r-generalized Fibonacci numbers are defined as*

$$F_n^{(r)} = \begin{cases} 0, & \text{for } n < r-1 \\ 1, & \text{for } n = r-1 \\ \sum_{i=1}^{r} F_{n-i}^{(r)}, & \text{for } n \geq r. \end{cases}$$

Note that the Fibonacci numbers are just $F_n^{(2)}$.

To state our identity, we need the following definition.

Definition 1.2. *Let $r \geq 1$ and $c \geq 0$ be integers. Define*

$$a_c^{(r)} = \sum_{i=c}^{r-1} \binom{i}{c} 2^{i-c}.$$

The first few rows and columns of the $a_c^{(r)}$ array are produced in the following table.

$r \backslash c$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	3	1										
3	7	5	1									
4	15	17	7	1								
5	31	49	31	9	1							
6	63	129	111	49	11	1						
7	127	321	351	209	71	13	1					
8	255	769	1023	769	351	97	15	1				
9	511	1793	2815	2561	1431	505	117	17	1			
10	1023	4097	7423	7937	5503	2561	799	161	19	1		
11	2047	9217	18943	23297	18943	10625	4159	1121	199	21	1	
12	4095	20481	47103	65537	61183	40193	18943	6401	1519	241	23	1

$a_c^{(r)}$ array

The following lemma gives an alternative definition of the $a_c^{(r)}$ array

Lemma 1.3. *Let $r \geq 1$ be an integer. Then*

$$a_0^{(r)} = 2^r - 1 \text{ and } a_{r-1}^{(r)} = 1.$$

Let $c \geq 1$ and $r \geq c+2$ be integers. Then

$$a_c^{(r)} = 2 \cdot a_c^{(r-1)} + a_{c-1}^{(r-1)}.$$

Proof. Let $r \geq 1$ be an integer. Then

$$a_0^{(r)} = \sum_{i=0}^{r-1} \binom{i}{0} 2^{i-0} = \sum_{i=0}^{r-1} 2^i = 2^r - 1$$

and

$$a_{r-1}^{(r)} = \sum_{i=r-1}^{r-1} \binom{i}{r-1} 2^{i-(r-1)} = \binom{r-1}{r-1} 2^{r-1-(r-1)} = 1.$$

Next, let $c \geq 1$ and $r \geq c+2$ be integers. Then

$$\begin{aligned}
2 \cdot a_c^{(r-1)} \quad + \quad a_{c-1}^{(r-1)} = 2 \cdot \sum_{i=c}^{r-2} \binom{i}{c} 2^{i-c} + \sum_{i=c-1}^{r-2} \binom{i}{c-1} 2^{i-c+1} \\
&= \sum_{i=c}^{r-2} \binom{i}{c} 2^{i-c+1} + \sum_{i=c-1}^{r-2} \binom{i}{c-1} 2^{i-c+1} \\
&= 1 + \sum_{i=c}^{r-2} \left(\binom{i}{c} + \binom{i}{c-1} \right) 2^{i-c+1} \\
&= 1 + \sum_{i=c}^{r-2} \binom{i+1}{c} 2^{i+1-c} \\
&= 1 + \sum_{i=c+1}^{r-1} \binom{i}{c} 2^{i-c} \\
&= \sum_{i=c}^{r-1} \binom{i}{c} 2^{i-c} \\
&= a_c^{(r)}.
\end{aligned}$$

This proves the lemma. □

We next state a technical lemma involving the generalized Fibonacci numbers.

Lemma 1.4. *Let* $1 \leq m \leq r$ *be a positive integer and* $n \geq mr$*. Then*

$$\begin{aligned}
F_n^{(r)} &= \sum_{i=1}^{m-1} (-1)^{i-1} a_{i-1}^{(r)} F_{n-ir}^{(r)} + (-1)^{m-1} \binom{r-1}{m-1} 2^{r-m} F_{n-mr}^{(r)} \\
&+ (-1)^{m-1} \sum_{k=1}^{r-1} \sum_{i=k}^{r-1} \binom{i-1}{m-1} 2^{i-m} F_{n-mr-k}^{(r)}.
\end{aligned}$$

Finally, the following corollary gives our identity for generalized Fibonacci numbers.

Corollary 1.5. *Let* $r \geq 1$ *and* $n \geq r^2$*. Then*

$$F_n^{(r)} = \sum_{i=0}^{r-1} (-1)^i a_i^{(r)} F_{n-r-ir}^{(r)}.$$

An example of our generalized Fibonacci number identity may be helpful. We let $r = 5$. The pentanacci numbers are displayed in the following table.

n	$F_n^{(5)}$	n	$F_n^{(5)}$	n	$F_n^{(5)}$	n	$F_n^{(5)}$	n	$F_n^{(5)}$
0	0	10	31	20	26784	30	23099186	40	19921290241
1	0	11	61	21	52656	31	45411804	41	39164225421
2	0	12	120	22	103519	32	89277256	42	76994839906
3	0	13	236	23	203513	33	175514464	43	151367869744
4	1	14	464	24	400096	34	345052351	44	297581396608
5	1	15	912	25	786568	35	678355061	45	585029621920
6	2	16	1793	26	1546352	36	1333610936	46	1150137953599
7	4	17	3525	27	3040048	37	2621810068	47	2261111681777
8	8	18	6930	28	5976577	38	5154342880	48	4445228523648
9	16	19	13624	29	11749641	39	10133171296	49	8739089177552

Pentanacci Numbers from 0 to 49

And our identity for the pentanacci numbers is

$$F_n^{(5)} = 31F_{n-5}^{(5)} - 49F_{n-10}^{(5)} + 31F_{n-15}^{(5)} - 9F_{n-20}^{(5)} + F_{n-25}^{(5)}.$$

In other words, we can compute a pentanacci number by calculating the alternating sum of the product of terms in the fifth row of the $a_c^{(r)}$ array with every fifth pentanacci number from the previous twenty-five.

2. Proof of Lemma 1.4

We will prove Lemma 1.4. Throughout the proof the integer $r \geq 2$ will be fixed. The proof is by induction on m. In the proof we will use known identities and perform substitutions and collections of terms.

For the base case of $m = 1$ we begin with the defining identity for the r-generalized Fibonacci numbers,

$$F_n^{(r)} = \sum_{i=1}^{r} F_{n-i}^{(r)}.$$

Breaking off $F_{n-1}^{(r)}$ and using the fact that $1 = \binom{0}{0}2^0$ we obtain

$$F_n^{(r)} = \binom{0}{0}2^0 F_{n-1}^{(r)} + \sum_{i=2}^{r} F_{n-i}^{(r)}.$$

Now replacing $F_{n-1}^{(r)}$ by $F_{n-2}^{(r)} + \cdots + F_{n-r-1}^{(r)}$ and collecting terms we obtain

$$F_n^{(r)} = \sum_{i=2}^{r} \left(1 + \binom{0}{0}2^0\right) F_{n-i}^{(r)} + \binom{0}{0}2^0 F_{n-r-1}^{(r)}.$$

Breaking off $F_{n-2}^{(r)}$ and using the fact that $1 + \binom{0}{0}2^0 = \binom{1}{0}2^1$ we obtain

$$F_n^{(r)} = \binom{1}{0}2^1 F_{n-2}^{(r)} + \sum_{i=3}^{r} \left(1 + \binom{0}{0}2^0\right) F_{n-i}^{(r)} + \binom{0}{0}2^0 F_{n-r-1}^{(r)}.$$

Now replacing $F_{n-2}^{(r)}$ by $F_{n-3}^{(r)} + \cdots + F_{n-r-2}^{(r)}$ and collecting terms we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=3}^{r} \left(1 + \binom{0}{0}2^0 + \binom{1}{0}2^1\right) F_{n-i}^{(r)} \\ &+ \left(\binom{0}{0}2^0 + \binom{1}{0}2^1\right) F_{n-r-1}^{(r)} + \binom{1}{0}2^1 F_{n-r-2}^{(r)}. \end{aligned}$$

Breaking off $F_{n-3}^{(r)}$ and using the fact that $1 + \binom{0}{0}2^0 + \binom{1}{0}2^1 = \binom{2}{0}2^2$ we obtain

$$\begin{aligned} F_n^{(r)} &= \binom{2}{0}2^2 F_{n-3}^{(r)} + \sum_{i=4}^{r} \left(1 + \binom{0}{0}2^0 + \binom{1}{0}2^1\right) F_{n-i}^{(r)} \\ &+ \left(\binom{0}{0}2^0 + \binom{1}{0}2^1\right) F_{n-r-1}^{(r)} + \binom{1}{0}2^1 F_{n-r-2}^{(r)}. \end{aligned}$$

Now replacing $F_{n-3}^{(r)}$ by $F_{n-4}^{(r)} + \cdots + F_{n-r-3}^{(r)}$ and collecting terms we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=4}^{r} \left(1 + \binom{0}{0}2^0 + \binom{1}{0}2^1 + \binom{2}{0}2^2\right) F_{n-i}^{(r)} \\ &+ \left(\binom{0}{0}2^0 + \binom{1}{0}2^1 + \binom{2}{0}2^2\right) F_{n-r-1}^{(r)} \\ &+ \left(\binom{1}{0}2^1 + \binom{2}{0}2^2\right) F_{n-r-2}^{(r)} + \binom{2}{0}2^2 F_{n-r-3}^{(r)}. \end{aligned}$$

Continuing this process for $F_{n-4}^{(r)}, \ldots, F_{n-r+1}^{(r)}$ we obtain

$$F_n^{(r)} = \left(1 + \sum_{i=0}^{r-2} \binom{i}{0} 2^i\right) F_{n-r}^{(r)} + \sum_{k=1}^{r-1}\sum_{i=k}^{r-1} \binom{i-1}{0} 2^{i-1} F_{n-r-k}^{(r)}.$$

But

$$1 + \sum_{i=0}^{r-2} \binom{i}{0} 2^i = \binom{r-1}{0} 2^{r-1}$$

so we have established the base case when $m = 1$.

Next, we assume the result for some $1 \leq m < r$ and prove the result for $m + 1$. Hence, we have that

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m-1} (-1)^{i-1} a_{i-1}^{(r)} F_{n-ir}^{(r)} + (-1)^{m-1} \binom{r-1}{m-1} 2^{r-m} F_{n-mr}^{(r)} \\ &+ (-1)^{m-1} \sum_{k=1}^{r-1}\sum_{i=k}^{r-1} \binom{i-1}{m-1} 2^{i-m} F_{n-mr-k}^{(r)}. \end{aligned}$$

Breaking off $F^{(r)}_{n-mr-1}$ we have

$$\begin{aligned} F^{(r)}_n &= \sum_{i=1}^{m-1}(-1)^{i-1}a^{(r)}_{i-1}F^{(r)}_{n-ir} + (-1)^{m-1}\binom{r-1}{m-1}2^{r-m}F^{(r)}_{n-mr} \\ &+ (-1)^{m-1}\sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m}F^{(r)}_{n-mr-1} \\ &+ (-1)^{m-1}\sum_{k=2}^{r-1}\sum_{i=k}^{r-1}\binom{i-1}{m-1}2^{i-m}F^{(r)}_{n-mr-k}. \end{aligned}$$

Replacing $F^{(r)}_{n-mr-1}$ by $F^{(r)}_{n-mr} - F^{(r)}_{n-mr-2} - F^{(r)}_{n-mr-3} - \cdots - F^{(r)}_{n-mr-r}$ we obtain

$$\begin{aligned} F^{(r)}_n &= \sum_{i=1}^{m-1}(-1)^{i-1}a^{(r)}_{i-1}F^{(r)}_{n-ir} \\ &+ (-1)^{m-1}\left(\binom{r-1}{m-1}2^{r-m} + \sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m}\right)F^{(r)}_{n-mr} \\ &+ (-1)^{m-1}\sum_{k=2}^{r-1}\sum_{i=k}^{r-1}\binom{i-1}{m-1}2^{i-m}F^{(r)}_{n-mr-k} \\ &+ (-1)^{m}\sum_{k=2}^{r}\sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m}F^{(r)}_{n-mr-k}. \end{aligned}$$

Simplifying the coefficient of $F^{(r)}_{n-mr}$ we have

$$\begin{aligned} &\binom{r-1}{m-1}2^{r-m} + \sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m} \\ = &\sum_{i=1}^{r}\binom{i-1}{m-1}2^{i-m} = a^{(r)}_{m-1}. \end{aligned}$$

So we have matched the first part of the formula for $F^{(r)}_n$ for $m+1$.

In addition, we observe that the coefficients of the $m-1$ terms $F^{(r)}_{n-mr-k}$ where $2 \le k \le m$ are 0. This is true because the quantity

$$(-1)^{m-1}\sum_{i=k}^{r-1}\binom{i-1}{m-1}2^{i-m} + (-1)^m\sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m}$$

consists of two sums of opposite sign which are identical except that the second sum contains

$$\sum_{i=1}^{k-1}\binom{i-1}{m-1}2^{i-m}.$$

But this sum is 0 since $2 \le k \le m$. The remaining terms in our identity involve multiples of $F^{(r)}_{n-mr-k}$ with $m+1 \le k \le r$. And the coefficients of $F^{(r)}_{n-mr-k}$ where

$k = m+1$ to $k = r-1$ are

$$(-1)^{m-1}\sum_{i=k}^{r-1}\binom{i-1}{m-1}2^{i-m} + (-1)^m\sum_{i=1}^{r-1}\binom{i-1}{m-1}2^{i-m} = (-1)^m\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}.$$

This last equality is true because the two sums are of opposite sign with the first sum contained in the second and the second sums' inital terms are 0 Furthermore, the coefficient of $F_{n-mr}^{(r)}$ matches the form of the first sum. Thus, our identity is

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\sum_{k=m+1}^{r}\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}F_{n-mr-k}^{(r)}. \end{aligned}$$

To complete our induction proof, we need to systematically eliminate the quantities $F_{n-mr-m-1}^{(r)}$ to $F_{n-mr-r+1}^{(r)}$ by making substitutions and collecting terms one by one. Breaking off $F_{n-mr-m-1}^{(r)}$ and using the fact that $\sum_{i=m}^{m}\binom{i-1}{m-1}2^{i-m} = \binom{m-1}{m-1}2^0 = \binom{m}{m}2^0$ we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\binom{m}{m}2^0F_{n-mr-m-1} + (-1)^m\sum_{k=m+2}^{r}\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}F_{n-mr-k}^{(r)}. \end{aligned}$$

Replacing $F_{n-mr-m-1}$ by $F_{n-mr-m-2} + F_{n-mr-m-3} + \cdots + F_{n-mr-m-r-1}$ and collecting terms we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\sum_{k=m+2}^{r}\left(\binom{m}{m}2^0 + \sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}\right)F_{n-mr-k} \\ &+ (-1)^m\sum_{k=r+1}^{r+1+m}\binom{m}{m}2^0F_{n-mr-k}^{(r)}. \end{aligned}$$

Breaking off $F_{n-mr-m-2}$ and using the fact that $\binom{m}{m}2^0 + \binom{m-1}{m-1}2^0 + \binom{m}{m-1}2^1 = \binom{m}{m}2^0 + \binom{m}{m}2^0 + \binom{m}{m-1}2^1 = \binom{m+1}{m}2^1$ we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\binom{m+1}{m}2^1F_{n-mr-m-2} \\ &+ (-1)^m\sum_{k=m+3}^{r}\left(\binom{m}{m}2^0+\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}\right)F_{n-mr-k} \\ &+ (-1)^m\sum_{k=r+1}^{r+1+m}\binom{m}{m}2^0F_{n-mr-k}^{(r)}. \end{aligned}$$

Now replacing $F_{n-mr-m-2}^{(r)}$ by $F_{n-mr-m-3}^{(r)}+\cdots+F_{n-mr-m-r-2}^{(r)}$ and collecting terms we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\sum_{k=m+3}^{r}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1+\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}\right)F_{n-mr-k}^{(r)} \\ &+ (-1)^m\sum_{k=r+1}^{r+1+m}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1\right)F_{n-mr-k}^{(r)} \\ &+ (-1)^m\binom{m+1}{m}2^1F_{n-mr-r-m-2}. \end{aligned}$$

Breaking off $F_{n-mr-m-3}^{(r)}$ and using the fact that $\binom{m}{m}2^0 + \binom{m-1}{m-1}2^0 + \binom{m}{m-1}2^1 + \binom{m+1}{m}2^1 + \binom{m+1}{m-1}2^2 = \binom{m+2}{m}2^2$ we obtain

$$\begin{aligned} F_n^{(r)} &= \sum_{i=1}^{m}(-1)^{i-1}a_{i-1}^{(r)}F_{n-ir}^{(r)} \\ &+ (-1)^m\binom{m+2}{m}2^2F_{n-mr-m-3} \\ &+ (-1)^m\sum_{k=m+4}^{r}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1+\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}\right)F_{n-mr-k} \\ &+ (-1)^m\sum_{k=r+1}^{r+1+m}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1\right)F_{n-mr-k}^{(r)} \\ &+ (-1)^m\binom{m+1}{m}2^1F_{n-mr-r-m-2}^{(r)}. \end{aligned}$$

Now replacing $F^{(r)}_{n-mr-m-3}$ by $F^{(r)}_{n-mr-m-4}+\cdots+F^{(r)}_{n-mr-r-m-3}$ and collecting terms we obtain

$$\begin{aligned}
F^{(r)}_n &= \sum_{i=1}^{m}(-1)^{i-1}a^{(r)}_{i-1}F^{(r)}_{n-ir}\\
+ & \quad (-1)^m \sum_{k=m+4}^{r}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1+\binom{m+2}{m}2^2+\sum_{i=m}^{k-1}\binom{i-1}{m-1}2^{i-m}\right)F^{(r)}_{n-mr-k}\\
+ & \quad (-1)^m \sum_{k=r+1}^{r+1+m}\left(\binom{m}{m}2^0+\binom{m+1}{m}2^1+\binom{m+2}{m}2^2\right)F^{(r)}_{n-mr-k}\\
+ & \quad (-1)^m\left(\binom{m+1}{m}2^1+\binom{m+2}{m}2^2\right)F^{(r)}_{n-mr-r-m-2}\\
+ & \quad (-1)^m\binom{m+2}{m}2^2F^{(r)}_{n-mr-r-m-3}.
\end{aligned}$$

Continuing this process for $F^{(r)}_{n-mr-m-4},\ldots,F^{(r)}_{n-mr-r+1}$ we obtain

$$\begin{aligned}
F^{(r)}_n &= \sum_{i=1}^{m}(-1)^{i-1}a^{(r)}_{i-1}F^{(r)}_{n-ir}\\
&+ (-1)^m\left(\sum_{i=m}^{r-2}\binom{i}{m}2^{i-m}+\sum_{i=m}^{r-1}\binom{i-1}{m-1}2^{i-m}\right)F^{(r)}_{n-mr-r}\\
&+ (-1)^m\sum_{k=r+1}^{r+1+m}\sum_{i=m}^{r-2}\binom{i}{m}2^{i-m}F^{(r)}_{n-mr-k}\\
&+ (-1)^m\sum_{k=r+m+2}^{2r-1}\sum_{i=k-r-1}^{r-2}\binom{i}{m}2^{i-m}F^{(r)}_{n-mr-k}.
\end{aligned}$$

But

$$\sum_{i=m}^{r-2}\binom{i}{m}2^{i-m}+\sum_{i=m}^{r-1}\binom{i-1}{m-1}2^{i-m}=\binom{r-1}{m}2^{r-m-1}$$

which matches the coefficient we wanted on the $F^{(r)}_{n-mr-r}$ term in the $m+1$ formula. At this point our formula reads

$$\begin{aligned}
F^{(r)}_n &= \sum_{i=1}^{m}(-1)^{i-1}a^{(r)}_{i-1}F^{(r)}_{n-ir}+(-1)^m\binom{r-1}{m}2^{r-m-1}F^{(r)}_{n-mr-r}\\
&+ (-1)^m\sum_{k=r+1}^{r+1+m}\sum_{i=m}^{r-2}\binom{i}{m}2^{i-m}F^{(r)}_{n-mr-k}\\
&+ (-1)^m\sum_{k=r+m+2}^{2r-1}\sum_{i=k-r-1}^{r-2}\binom{i}{m}2^{i-m}F^{(r)}_{n-mr-k}.
\end{aligned}$$

Replacing i by $i-1$ in the last two sums we have

$$\begin{aligned}
& (-1)^m \sum_{k=r+1}^{r+1+m} \sum_{i=m+1}^{r-1} \binom{i-1}{m} 2^{i-(m+1)} F^{(r)}_{n-mr-k} \\
+ \quad & (-1)^m \sum_{k=r+m+2}^{2r-1} \sum_{i=k-r}^{r-1} \binom{i-1}{m} 2^{i-(m+1)} F^{(r)}_{n-mr-k}.
\end{aligned}$$

And replacing k by $k+r$ in these two sums we have

$$\begin{aligned}
& (-1)^m \sum_{k=1}^{m+1} \sum_{i=m+1}^{r-1} \binom{i-1}{m} 2^{i-(m+1)} F^{(r)}_{n-(m+1)r-k} \\
+ \quad & (-1)^m \sum_{k=m+2}^{r-1} \sum_{i=k}^{r-1} \binom{i-1}{m} 2^{i-(m+1)} F^{(r)}_{n-(m+1)r-k}.
\end{aligned}$$

Finally, replacing $m+1$ (the lower limit on i in the first inner sum) by k, since these terms are 0, and combining the two sums into one we have the sum

$$(-1)^m \sum_{k=1}^{r-1} \sum_{i=k}^{r-1} \binom{i-1}{m} 2^{i-m-1} F^{(r)}_{n-mr-r-k}$$

which is the final term we needed to match-up. This makes the result true for $m+1$. Therefore, by mathematical induction, the result is true for $1 \le m \le r$.

3. Proof of Corollary

Applying Lemma 1.4 with $m = r$ we have

$$\begin{aligned}
F^{(r)}_n \quad = \quad & \sum_{i=1}^{r-1} (-1)^{i-1} a^{(r)}_{i-1} F^{(r)}_{n-ir} \\
+ \quad & (-1)^{r-1} F^{(r)}_{n-r^2} \\
+ \quad & (-1)^{r-1} \sum_{k=1}^{r-1} \sum_{i=k}^{r-1} \binom{i-1}{r-1} 2^{i-r} F^{(r)}_{n-r^2-k}.
\end{aligned}$$

But $a^{(r)}_{r-1} = 1$ so $(-1)^{r-1} F^{(r)}_{n-r^2} = (-1)^{r-1} a^{(r)}_{r-1} F^{(r)}_{n-r^2}$ and the last double sum is 0. The corollary follows.

References

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