Two Identities Involving Generalized Fibonacci Numbers

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Abstract. Let $r \geq 2$ be an integer. The r-generalized Fibonacci sequence $\{G_n\}$ is defined as

$$G_n = \begin{cases} 0, & \text{for } 0 \leq n < r-1 \\ 1, & \text{for } n = r-1 \\ \sum_{i=1}^{r} G_{n-i}, & \text{for } n \geq r. \end{cases}$$

We will present two identities involving r-generalized Fibonacci numbers.

1. Introduction

Several generalizations of Fibonacci numbers and identities have been studied by mathematicians over the years.

For example, Melham and Shannon [5] let a, b, p, and q be real numbers. They then defined a generalized Fibonacci sequence $\{W_n\}$, where $W_0 = a$, $W_1 = b$, and for $n \geq 2$, $W_n = pW_{n-1} - qW_{n-2}$. Finally, they showed that for $n \geq 0$,

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3W_{n+2} - q^2W_{n+1}),$$

where $e = pab - qa^2 - b^2$.

In another example, Howard [3] defined a generalized Tribonacci sequence $\{V_n\}$, where V_0, V_1, and V_2 are arbitrary complex numbers, and r, s, and t are arbitrary integers, with $t \neq 0$ and for $n \geq 3$ as

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}.$$

The sequence can be extended in the usual way to negative subscripts. Using the notation

$$V_n = V_n(V_0, V_1, V_2; r, s, t)$$

to emphasize the initial conditions and coefficients, he also defined the sequence $\{J_n\}$, where

$$J_n = V_n(3, r, r^2 + 2s; r, s, t).$$

Howard then proved the identity

$$V_{n+2m} = J_m V_{n+m} - t^m J_{-m} V_n + t^m V_{n-m},$$

where n and m are arbitrary integers.

In this paper we define another generalization of the Fibonacci sequence we call the r-generalized Fibonacci sequence, where $r \geq 2$ is an integer. This definition and an identity for the r-generalized Fibonacci sequence were given in [2]. We will then state and prove two identities involving the r-generalized Fibonacci sequence.

Definition 1.1. *Let $r \geq 2$ be an integer. The r-generalized Fibonacci sequence $\{G_n\}$ is defined as*

$$G_n = \begin{cases} 0, & \text{for } 0 \leq n < r-1 \\ 1, & \text{for } n = r-1 \\ \sum_{i=1}^{r} G_{n-i}, & \text{for } n \geq r. \end{cases}$$

Note that the Fibonacci sequence, $\{F_n\}$, is just the 2-generalized Fibonacci sequence. The first few terms of the r-generalized Fibonacci sequence for $2 \leq r \leq 8$ are given in the following table.

r-generalized Fibonacci Sequences

$r \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136
4	0	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872
5	0	0	0	0	1	1	2	4	8	16	31	61	120	236	464	912	1793
6	0	0	0	0	0	1	1	2	4	8	16	32	63	125	248	492	976
7	0	0	0	0	0	0	1	1	2	4	8	16	32	64	127	253	504
8	0	0	0	0	0	0	0	1	1	2	4	8	16	32	64	128	255

The r-generalized Fibonacci sequences for $r = 2, 3, 4, 5, 6, 7, 8$ can be found in Sloane [6] as sequences A000045, A000073, A000078, A001591, A001592, A122189, and A079262, respectively.

We note that for each r-generalized Fibonacci sequence $\{G_n\}$,

$$G_n = 2^{n-r} \quad \text{for } r \leq n \leq 2r - 1.$$

2. First Identity

The following theorem gives our first identity involving the r-generalized Fibonacci sequence.

Theorem 2.1. *Let $r \geq 2$ be an integer, $\{G_n\}$ be the r-generalized Fibonacci sequence, and $n \geq 2r - 1$ be an integer. Then*

$$G_n = 2^{r-1} G_{n-r} + \sum_{k=1}^{r-1} \left(\sum_{i=k}^{r-1} 2^{i-1} \right) G_{n-r-k}.$$

For the Fibonacci sequence, this identity is

$$F_n = 2F_{n-2} + F_{n-3}.$$

Listing this identity for $r = 2, 3, 4, 5,$ and 6 we have the resulting formulas.

$$\begin{aligned}
r = 2: \quad G_n &= 2G_{n-2} + G_{n-3} \\
r = 3: \quad G_n &= 4G_{n-3} + 3G_{n-4} + 2G_{n-5} \\
r = 4: \quad G_n &= 8G_{n-4} + 7G_{n-5} + 6G_{n-6} + 4G_{n-7} \\
r = 5: \quad G_n &= 16G_{n-5} + 15G_{n-6} + 14G_{n-7} + 12G_{n-8} + 8G_{n-9} \\
r = 6: \quad G_n &= 32G_{n-6} + 31G_{n-7} + 30G_{n-8} + 28G_{n-9} + 24G_{n-10} + 16G_{n-11}.
\end{aligned}$$

We will give two proofs of Theorem 2.1. The first proof uses basic algebra.

Proof. We start with G_n and replace it with $G_{n-1} + G_{n-2} + \cdots + G_{n-r}$. In this expression we replace G_{n-1} with $G_{n-2} + G_{n-3} + \cdots + G_{n-r-1}$ and collect like terms. In this expression we replace G_{n-2} with $G_{n-3} + G_{n-4} + \cdots + G_{n-r-2}$ and collect like terms. We continue this process a total of r times. The last term we replace is G_{n-r+1} and it is replaced by $G_{n-r} + G_{n-r-1} + \cdots + G_{n-2r+1}$. We display this process with the following set of equations.

$$\begin{aligned}
G_n &= G_{n-1} + G_{n-2} + \cdots + G_{n-r} \\
&= (1+1)G_{n-2} + (1+1)G_{n-3} + \cdots + (1+1)G_{n-r} + G_{n-r-1} \\
&= (1+1+2)G_{n-3} + (1+1+2)G_{n-4} + \cdots + (1+1+2)G_{n-r} \\
&\quad +(1+2)G_{n-r-1} + 2G_{n-r-2} \\
&= (1+1+2+4)G_{n-4} + \cdots + (1+1+2+4)G_{n-r} + (1+2+4)G_{n-r-1} \\
&\quad +(2+4)G_{n-r-2} + 4G_{n-r-3} \\
&= \cdots \\
&= (1+1+2+4+\cdots+2^{r-2})G_{n-r} + (1+2+4+\cdots+2^{r-2})G_{n-r-1} \\
&\quad +(2+4+\cdots+2^{r-2})G_{n-r-2} + \cdots + 2^{r-2}G_{n-2r-1}.
\end{aligned}$$

But this is what we wanted to prove. □

The second proof is a combinatorial proof.

Proof. Let $\{u_n\}$ be the sequence defined by $u_n = G_{n+r-1}$ for $n \geq 0$. Then, according to [1, pp. 36,4], u_n counts the number of tilings of an n-board with tiles of length at most r. For convenience, we call a tiling of an n-board with tiles of length at most r an n-tiling. We first note that for $1 \leq i \leq r$, $u_i = 2^{i-1}$. That is, the number of i-tilings for $1 \leq i \leq r$ is 2^{i-1}. To prove Theorem 2.1 combinatorially, we will prove the following statement. Let $n \geq r$. Then

$$u_n = 2^{r-1}u_{n-r} + \sum_{k=1}^{r-1}\left(\sum_{i=k}^{r-1} 2^{i-1}\right) u_{n-r-k}.$$

We prove this statement by answering the following question in two ways.

Question. How many n-tilings are there?

Answer 1. By definition, there are u_n n-tilings.

Answer 2. Because the tiles are of length $1, 2, \ldots, r$, every n-tiling has at least one break between cells $n-2r+1, \ldots, n-r$. Put each n-tiling in one of r disjoint classes according to the largest $n - 2r + 1 \leq i \leq n - r$ where the n-tiling has a break at cell i. We can count the number of n-tilings whose largest break is at cell $n - r$ by multiplying the number of r-tilings times the number of $(n-r)$-tilings. The number of r-tilings is 2^{r-1} and the number of $(n - r)$-tilings is u_{n-r}. So the number of n-tilings whose largest break is at cell $n-r$ is $2^{r-1} \cdot u_{n-r}$. Next, to count the number of n-tilings where the largest cell where there is a break is at cell $n-r-1$, we note that for any of these n-tilings, the next tile after the break at cell $n - r - 1$ is of length $2, 3, \ldots, r$. And after this tile of length $2, 3, \ldots, r$, the number of cells left to tile is $r - 1$, $r - 2$, …, 1 and the number of k-tilings for $r - 1 \geq k \geq 1$ is $2^{r-2}, 2^{r-3}, \ldots, 1$, respectively. And the number of $(n - r - 1)$-tilings is u_{n-r-1}. So the number of n-tilings whose largest break is at cell $n - r - 1$ is $(2^{r-2} + 2^{r-3} + \cdots + 1) \cdot u_{n-r-1}$, In general, we need to count the number of n-tilings where the largest cell where there is a break is at cell $n-r-i$, where $0 \leq i \leq r-1$. The next tile in these n-tiling is of length $i + 1, i + 2, \ldots, r$. And after this tile of length $i + 1, i + 2, \ldots, r$, the number of cells left to tile is $r-1$, …, i and the number of k-tilings for $r-1 \geq k \geq i$ is 2^{r-2}, …, 2^{i-1} respectively. And the number of $(n-r-i)$-tilings is u_{n-r-i}. So the number of n-tilings whose largest break is at cell $n - r - i$ is $(2^{r-2} + \cdots + 2^{i-1}) \cdot u_{n-r-i}$. Adding each of these terms completes the count. □

3. Second Identity

Theorem 3.1. *Let $r \geq 2$ be an integer, $\{G_n\}$ be the r-generalized Fibonacci sequence, and $n \geq 0$ be an integer. Then*

$$\sum_{k=0}^{n} G_k^2 + \sum_{i=2}^{r-1} \sum_{k=0}^{n-i} G_k G_{k+i} = G_n G_{n+1}.$$

The special case of this identity for the Fibonacci sequence was discovered by Lucas in 1876. He discovered that for $n \geq 0$,

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Its proof can be found in [4, pp. 77–78].

We will prove the second identity by induction on n.

Proof. The proof is by induction on n.

The base case of $n = 0$ is immediate since both sides of the identity are 0. Now assume the theorem is true for some $n \geq 0$. Then

$$\begin{aligned}
& \sum_{k=0}^{n+1} G_k^2 + \sum_{i=2}^{r-1} \sum_{k=0}^{n+1-i} G_k G_{k+i} \\
= & \sum_{k=0}^{n} G_k^2 + \sum_{i=2}^{r-1} \sum_{k=0}^{n-i} G_k G_{k+i} \\
+ & G_{n+1}^2 + \sum_{i=2}^{r-1} G_{n+1-i} G_{n+1} \\
= & G_n G_{n+1} + G_{n+1}^2 + G_{n+1} \sum_{i=2}^{r-1} G_{n+1-i} \\
= & G_{n+1} \sum_{i=0}^{r-1} G_{n+1-i} \\
= & G_{n+1} G_{n+2}.
\end{aligned}$$

This is the statement of the theorem for $n + 1$. Therefore, by induction, the theorem is true.

□

References

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