

To Your Hearts' Content

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Arthur Benjamin [1] showed the following amazing mathematical card trick during the 2007 Annual meeting of the AMS/MAA in New Orleans. He asked a volunteer from the audience to shuffle 20 high cards from a standard deck of playing cards. Then he arranged the deck, with “cooperation” from the volunteer, into a pile and then dealt-out the cards in a 4×5 array with some cards facing up and some cards facing down. After folding the cards inwards row-by-row or column-by-column several times, based on the volunteer’s choices, he piled all the cards into a single stack. When he spread the cards out on the table, all cards except five were facing up. Magically, the five hidden cards were all hearts. Benjamin then gave an explanation of the trick using parity arguments. We are going to use matrix multiplication and parity ideas to explain why the trick works, and then we’ll explain the mechanics of actually performing it, so the reader can use it to amaze their friends and neighbors.

The Face Matrix and The Folding Operations

In order to make the card trick easier to analyze, we introduce a *face matrix* to keep track of the orientations (face up or face down) of the cards. In a face matrix, entries will be 1’s and 0’s. The 1’s indicate face-up cards and 0’s indicate face-down cards. The following shows an arrangement of 20 high cards and the corresponding face matrix. The Jack of Hearts and the Ten of Hearts are placed face down at positions (2, 1) and (2, 5), respectively.

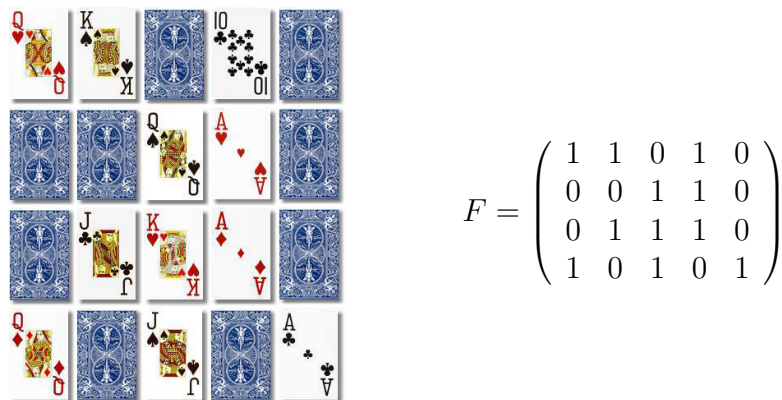


Figure 1 An arrangement of 20 high cards and its face matrix.

When we fold one row or one column of cards on top of the next row or next column, we are doing two operations. One is the flipping of all the cards in the row or column from face-up to face-down or vice-versa. The other operation is physically moving the cards inwards on top of the next row or next column. However, this

second operation is hard to implement with the face matrix since we would have more than one layer of cards in one row or column. Therefore, we shall flip the cards only, and leave the cards in their original positions. While it is convenient to ignore the reordering of the cards by leaving them in their original position, we lose no important information about the trick, since we only care about which card are face up and which are face down. Throughout our discussion, our matrices will continue to keep track of the orientations of the cards, based on their starting position. That is, if at some later time the $(1, 1)$ entry of our current matrix is 0, then the card that was originally in position $(1, 1)$ is face-down, though it may no longer be in position $(1, 1)$.

To illustrate how we use the face matrix, suppose that, starting with the arrangement in Fig. 1, we first fold the first row onto the second row. Then fold the second row onto the third row. The cards in the first row have been flipped twice while the cards in the second row are flipped once. Note that this flipping operation is equivalent to multiplying the following two operator matrices to left of the face matrix,

$$\begin{aligned} & \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where the t operator flips the cards from face-up to face-down and from face-down to face-up. Namely, when multiplying t and an entry of a face matrix, $t1 = 1t = 0$ and $t0 = 0t = 1$. When we multiply two operator matrices, $tt = 1$.

In general, inward row-folding operations and column-folding operations can be represented using matrices. For an $m \times n$ face matrix, there are two such row-folding operators

$$\begin{aligned} R_i^+ &= \begin{pmatrix} tI_i & 0 \\ 0 & I_{m-i} \end{pmatrix}_{m \times m} \quad \text{for } i < m \text{ and} \\ R_i^- &= \begin{pmatrix} I_{i-1} & 0 \\ 0 & tI_{m-i+1} \end{pmatrix}_{m \times m} \quad \text{for } i > 1 \end{aligned}$$

to fold row i onto row $i + 1$ and row i onto row $i - 1$, respectively. There are also two such column-folding operators

$$\begin{aligned} C_j^+ &= \begin{pmatrix} tI_j & 0 \\ 0 & I_{n-j} \end{pmatrix}_{n \times n} \quad \text{for } j < n \text{ and} \\ C_j^- &= \begin{pmatrix} I_{j-1} & 0 \\ 0 & tI_{n-j+1} \end{pmatrix}_{n \times n} \quad \text{for } j > 1 \end{aligned}$$

to fold column j onto column $j + 1$ and column j onto column $j - 1$, respectively. These operator matrices are diagonal matrices, hence the multiplication among them is commutative.

We define a (p, q) -folding to be a sequence of folding operations so that all the cards are piled up at the (p, q) -position. Since the C 's and the R 's commute, and

matrix multiplication is associative, any (p, q) -folding of the face matrix F can be represented by the product of matrices:

$$R_1^+ \cdots R_{p-1}^+ R_{p+1}^- \cdots R_m^- F C_1^+ \cdots C_{q-1}^+ C_{q+1}^- \cdots C_n^-.$$

We denote

$$R_p = R_1^+ \cdots R_{p-1}^+ R_{p+1}^- \cdots R_m^-, \text{ and } C_q = C_1^+ \cdots C_{q-1}^+ C_{q+1}^- \cdots C_n^-.$$

It is not difficult to verify that R_p and C_q depend only on the parities of p and q , in the way we describe next.

LEMMA. R_p and C_q are diagonal matrices: $R_p = \text{diag}(t, 1, t, 1, \dots)$ for even p and $R_p = \text{diag}(1, t, 1, t, \dots)$ for odd p , and $C_q = \text{diag}(t, 1, t, 1, \dots)$ for even q and $C_q = \text{diag}(1, t, 1, t, \dots)$ for odd q .

Based on the arrangement in Fig. 1, we perform the following $(3, 1)$ -folding: fold row 1 onto row 2, row 4 onto row 3, row 2 onto row 3, column 5 onto column 4, column 4 onto column 3, column 3 onto column 2, column 2 onto column 1. These folding operations result in all hearts face up and all non-hearts face down. The result is represented by the following equation.

$$R_1^+ R_2^+ R_4^- F C_2^- C_3^- C_4^- C_5^- = R_3 F C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

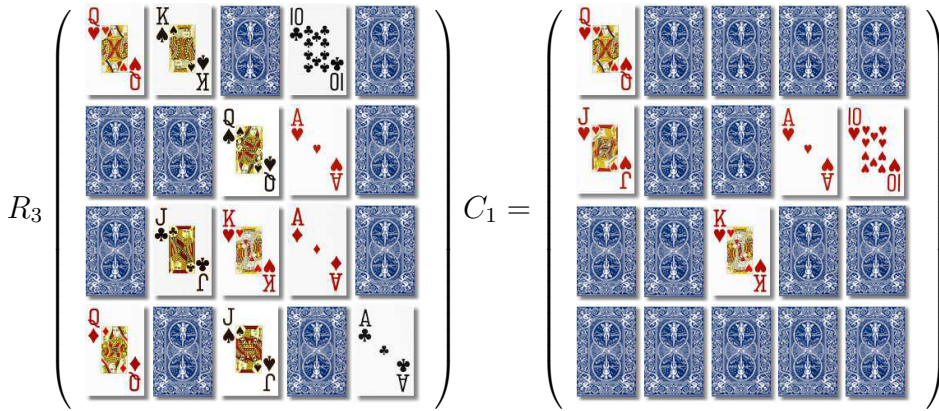


Figure 2 All hearts face up after a $(3, 1)$ -folding.

Folding and Arrangement Parities

This magic card trick is all about parity. Cards precisely arranged will produce the trick when folded inwards to a single pile. To explain these arrangements and folding, we will define some parities. The parity of position (i, j) in an arrangement is defined as the parity of $i + j$. The parity of a (p, q) -folding is defined as the parity of $p + q$.

For many arrangements of our cards, no folding will result in having exactly the hearts face down. Only the following two arrangements of cards allow the magic to

occur. One such arrangement of cards is called an *even arrangement* in which face-up hearts and face-down non-hearts are placed at even positions while face-down hearts and face-up non-hearts are at odd positions. The other arrangement of cards is called an *odd arrangement* in which face-up hearts and face-down non-hearts are placed at odd positions while face-down hearts and face-up non-hearts are at even positions.

The following theorem tells which folding flips which cards.

THEOREM. *A (p, q) -folding flips a card if and only if the card's position parity is opposite to the folding parity.*

Proof. Let $R_p F C_q = F'$, where face matrices $F = (f_{ij})_{m \times n}$ and $F' = (f'_{ij})_{m \times n}$. Applying the Lemma, if (i, j) and (p, q) have the same parity, then $f'_{ij} = 1f_{ij}1 = f_{ij}$ or $f'_{ij} = tf_{ij}t = tt f_{ij} = f_{ij}$. This means after the (p, q) -folding, the card at position (i, j) is not flipped. However, if (i, j) and (p, q) have opposite parities, then $f'_{ij} = tf_{ij}1 = tf_{ij}$ or $f'_{ij} = 1f_{ij}t = tf_{ij}$, namely this card is flipped. ■

COROLLARY 1. *If the arrangement and the folding have the same parity, then the resulting array of cards has all hearts face up and all non-hearts face down.*

COROLLARY 2. *If the arrangement and the folding have the opposite parities, then the resulting array of cards has all hearts face down and all non-hearts face up.*

Remember if, after folding, all the hearts face up, you should quickly flip the whole pile of cards over so that all the hearts are hidden.

Making an Even or Odd Arrangement of Cards

We see now that the cards in Fig. 1 form an even arrangement. To perform the trick, we need to produce an even (or odd) arrangement on the fly, without the audience realizing what is going on. To describe how to do this, suppose we're going to make an $m \times n$ arrangement of cards. Shuffle the $m \times n$ cards. From the top pick up two cards at a time. Ask the volunteer to decide which of the two cards should be placed on top. Then stack them up in one pile.

There are three cases we need to deal with:

1. If the two cards are hearts, then we place them back-to-back. We put the volunteer's card on top.
2. If the two cards are non-hearts, then we place them face-to-face. We put the volunteer's card on top.
3. One card is a heart and the other is a non-heart. If the volunteer wants the heart on top, then we place both cards face up with the heart on top. If the volunteer wants the non-heart on top, we place both cards face down with the heart on the bottom.

Now we wish to deal the cards from the top of the stack into an $m \times n$ array. In order to achieve an even arrangement, we need to consider whether n is odd or even.

If n is odd, there are many ways to deal the pile to an even arrangement. We show three ways of dealing the cards. Certain combinations of the three methods below will result in an even arrangement too.

1. Place n cards from left-to-right in each row. Make m rows from the top down. In this placement, the first card from the pile will be placed in row 1, column

1, which is of even position parity. According to the placement of the cards in the pile, if the card is a heart, it faces up; if the card is a non-heart, it faces down. In fact, at any even position, this is always the case. The card in row 1, column 2, which has an odd position parity, will be the second card in the pile. If the card is a heart, it faces down; if the card is a non-heart, it faces up. In fact, at any odd position, this is always the case.

2. Place n cards from left-to-right in odd rows and right-to-left in even rows. Make m rows from the top down.
3. Spiral placement. Place the cards from outside in: starting from the top row deal from left-to-right, the last column from top-to-bottom, the bottom row from right-to-left, the first column from bottom-to-top, etc.

If n is even, the first method for the odd n case will not work. However, the other two methods will result in an even arrangement.

To make an odd arrangement, based on the above technique for even arrangement, we simply interchange the orientations of hearts and non-hearts.

Some Examples

In the first example, we choose 12 cards that are all hearts. Using an even arrangement, we put them in a 4×3 array. A $(2, 2)$ -folding results in all cards (hearts) face up. According to the matrix representation, we have the following equation:

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For another example we show the magic also works when cards are placed in an incomplete array. However, it is harder to deal them in an even or odd arrangement.

We remove 9 cards from the arrangement in Fig. 2 to form an incomplete 4×5 array in an even arrangement. The corresponding face matrix for this incomplete array is also an incomplete matrix with spaces at the positions where no cards are placed. When you do the matrix multiplication, simply ignore the spaces. Then apply a $(3, 4)$ -folding which is odd. The resulting array of cards will have only non-hearts facing up.

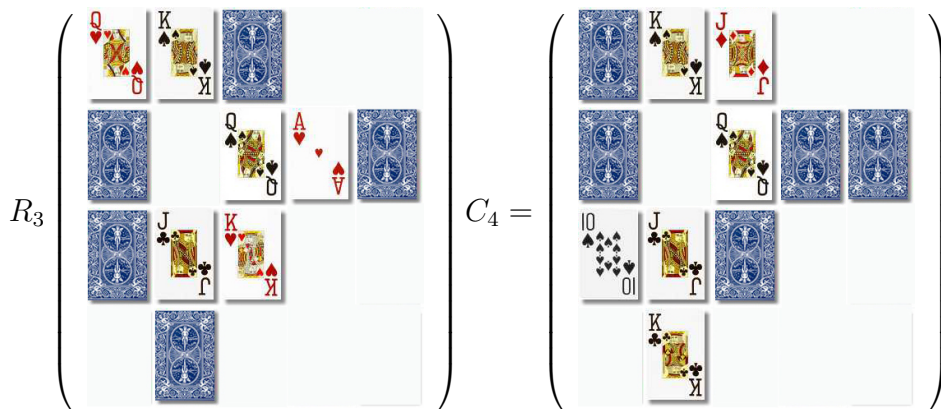


Figure 3 A $(3, 4)$ -folding turns non-hearts face up and hearts hidden.

Now, perform this amazing mathematical card trick to your hearts' content.

REFERENCES

1. A. T. Benjamin, "An Amazing Mathematical Card Trick," Abstracts of the Annual Meeting of the AMS/MAA, New Orleans, LA, 2007.