SOME IDENTITIES INVOLVING DIFFERENCES OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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ABSTRACT. Melham discovered the Fibonacci identity

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.$$

Melham then considered the generalized sequence W_n where $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ and a, b, p and q are integers and $q \neq 0$. Letting $e = pab - qa^2 - b^2$, he then stated and proved the following identity.

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1}).$$

There are similar differences of products of Fibonacci numbers like this one discovered by Fairgrieve and Gould.

$$F_n F_{n+4} F_{n+5} - F_{n+3}^3 = (-1)^{n+1} F_{n+6}.$$

We will discover, generalize, and prove similar identities. For example, a generalization of Fairgrieve and Gould's identity is

$$W_n W_{n+4} W_{n+5} - W_{n+3}^3 = eq^n (p^3 W_{n+4} - qW_{n+5}).$$

1. Introduction and Results

Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Many authors have studied Fibonacci identities and generalized Fibonacci identities. For example, Fairgrieve and Gould [1], Hoggatt and Bergum [2], and Horadam [3] stated and prove Fibonacci identities involving differences of products of Fibonacci numbers. And Melham [4] found, proved, and generalized the following Fibonacci identity involving differences of products of Fibonacci numbers.

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.$$

We will attempt to discover and prove some more differences of products of Fibonacci and generalized Fibonacci identities numbers.

The following is the sequence Melham used to generalize his Fibonacci identity involving differences of products of Fibonacci numbers.

Definition 1. *Let W_n be defined by $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$ and a, b, p and q are integers and $q \neq 0$. Let $e = pab - qa^2 - b^2$.*

Using this sequence, Melham proved the following identity.

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1}).$$

We now list some known and some new identities involving differences of products of generalized Fibonacci numbers. We will identify which identities are known and which are new. We will give some proofs of the new ones.

1a	$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n$
1b	$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1})$
2a	$F_n F_{n+4} F_{n+5} - F_{n+3}^3 = (-1)^{n+1} F_{n+6}$
2b	$W_n W_{n+4} W_{n+5} - W_{n+3}^3 = eq^n (p^3 W_{n+4} - q W_{n+5})$
3a	$F_n F_{n+3}^2 - F_{n+2}^3 = (-1)^{n+1} F_{n+1}$
3b	$W_n W_{n+3}^2 - W_{n+2}^3 = eq^n (p W_{n+3} + q W_{n+2})$
4a	$F_n^2 F_{n+3} - F_{n+1}^3 = (-1)^{n+1} F_{n+2}$
4b	$W_n^2 W_{n+3} - W_{n+1}^3 = eq^n (p W_n + W_{n+1})$
5a	$F_n F_{n+5} F_{n+6} - F_{n+3} F_{n+4}^2 = (-1)^{n+1} L_{n+6}$
5b	$W_n W_{n+5} W_{n+6} - W_{n+3} W_{n+4}^2 = eq^n (p W_{n+8} + p^3 q W_{n+4})$
6a	$F_n F_{n+4}^2 - F_{n+2} F_{n+3}^2 = (-1)^{n+1} L_{n+3}$
6b	$W_n W_{n+4}^2 - W_{n+2} W_{n+3}^2 = eq^n (p^2 W_{n+4} + q^2 W_{n+2})$
7a	$F_n F_{n+3} F_{n+5} - F_{n+2}^2 F_{n+4} = (-1)^{n+1} L_{n+2}$
7b	$W_n W_{n+3} W_{n+5} - W_{n+2}^2 W_{n+4} = eq^n (p^2 W_{n+4} + q^3 W_n)$
8a	$F_n F_{n+3}^2 - F_{n+1}^2 F_{n+4} = (-1)^{n+1} L_{n+2}$
8b	$W_n W_{n+3}^2 - W_{n+1}^2 W_{n+4} = eq^n (W_{n+4} - q^2 W_n)$
9a	$F_n F_{n+2} F_{n+5} - F_{n+1} F_{n+3}^2 = (-1)^{n+1} L_{n+3}$
9b	$W_n W_{n+2} W_{n+5} - W_{n+1} W_{n+3}^2 = eq^n (W_{n+5} + p^2 q W_{n+1})$
10	$F_n F_{n+2} F_{n+4} F_{n+6} - F_{n+3}^4 = (-1)^{n+1} L_{n+3}^2$
11	$F_n F_{n+4}^3 - F_{n+2}^3 F_{n+6} = (-1)^{n+1} F_{n+3} L_{n+3}$
12	$F_n^2 F_{n+5}^3 - F_{n+1}^3 F_{n+6}^2 = (-1)^{n+1} L_{n+3}^3$

Identities 1a and 1b were discovered and proved by Melham [4]. Identity 2a was discovered and proved by Fairgrieve and Gould [1]. Identities 3a, 4a, and 8a were discovered and proved by Hoggatt and Bergum [2]. As far as we know, the other identities in the table are new. In the next section, we will prove some of the new identities. The proofs of all the generalized identities are similar to the proof of 1b by Melham [4].

2. Some Proofs

We first present the proof of identity 2b.

Proof of 2b. To prove 2b, we require the identity

$$W_nW_{n+2} - W_{n+1}^2 = eq^n.$$

This identity was proved by Horadam [3, p. 171, eq. (4.3)]. We also need the following identities

$$\begin{aligned}
W_{n+2} &= pW_{n+1} - qW_n,\\
W_{n+3} &= (p^2 - q)W_{n+1} - pqW_n,\\
W_{n+4} &= (p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n,\\
W_{n+5} &= (p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n.
\end{aligned}$$

These identities are obtained by the use of the recurrence for W_n. To prove 2b, we write the LHS and RHS of identity 2b in terms of W_n, W_{n+1}, p and q. The RHS of 2b is

$$\begin{aligned}
&eq^n(p^3W_{n+4} - qW_{n+5})\\
&= (W_nW_{n+2} - W_{n+1}^2)(p^3W_{n+4} - qW_{n+5})\\
&= (W_n(pW_{n+1} - qW_n) - W_{n+1}^2)\\
&\times \left(p^3\left((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\right) - q\left((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\right)\right)\\
&= (p^7 - 2p^5q + p^3q^2 + pq^3)W_{n+1}^2W_n + (-2p^6q + 5p^4q^2 - 5p^2q^3 + q^4)W_{n+1}W_n^2\\
&\quad + (-p^6 + 3p^4q - 3p^2q^2 + q^3)W_{n+1}^3 + (p^5q^2 - 2p^3q^3 + 2pq^4)W_n^3.
\end{aligned}$$

The LHS of 2b is

$$\begin{aligned}
&W_nW_{n+4}W_{n+5} - W_{n+3}^2\\
&= W_n\left((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\right)\left((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\right)\\
&\quad - \left((p^2 - q)W_{n+1} - pqW_n\right)^3\\
&= (p^7 - 2p^5q + p^3q^2 + pq^3)W_{n+1}^2W_n + (-2p^6q + 5p^4q^2 - 5p^2q^3 + q^4)W_{n+1}W_n^2\\
&\quad + (-p^6 + 3p^4q - 3p^2q^2 + q^3)W_{n+1}^3 + (p^5q^2 - 2p^3q^3 + 2pq^4)W_n^3.
\end{aligned}$$

Since the LHS and RHS of identity 2b are equal, the identity is proved. □

We next present the proof of identity 5b.

Proof of 5b. To prove 5b, we again require the Horadam identity

$$W_nW_{n+2} - W_{n+1}^2 = eq^n.$$

We also need the following identities

$$\begin{aligned}
W_{n+2} &= pW_{n+1} - qW_n, \\
W_{n+3} &= (p^2 - q)W_{n+1} - pqW_n, \\
W_{n+4} &= (p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n, \\
W_{n+5} &= (p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n, \\
W_{n+6} &= (p^5 - 4p^3q + 3pq^2)W_{n+1} - (p^4q - 3p^2q^2 + q^3)W_n, \\
W_{n+8} &= (p^7 - 6p^5q + 10p^3q^2 - 4pq^3)W_{n+1} - (p^6q - 5p^4q^2 + 6p^2q^3 - q^4)W_n.
\end{aligned}$$

These identities are obtained by the use of the recurrence for W_n. To prove 5b, we write the LHS and RHS of identity 5b in terms of W_n, W_{n+1}, p and q. The RHS of 5b is

$$\begin{aligned}
& eq^n(pW_{n+8} + p^3qW_{n+4}) \\
& = (W_nW_{n+2} - W_{n+1}^2)(pW_{n+8} + p^3qW_{n+5}) \\
& = (W_n(pW_{n+1} - qW_n) - W_{n+1}^2) \\
& \times (p\left((p^7 - 6p^5q + 10p^3q^2 - 4pq^3)W_{n+1} - (p^6q - 5p^4q^2 + 6p^2q^3 - q^4)W_n\right) \\
& + p^3q\left((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\right)) \\
& = W_n^3(-pq^5 + 5p^3q^4 - 4p^5q^3 + p^7q^2) \\
& \quad W_n^2W_{n+1}(5p^2q^4 - 13p^4q^3 + 9p^6q^2 - 2p^8q) \\
& \quad W_nW_{n+1}^2(-pq^4 + p^3q^3 + 4p^5q^2 - 4p^7q + p^9) \\
& \quad W_{n+1}^3(4p^2q^3 - 8p^4q^2 + 5p^6q - p^8).
\end{aligned}$$

The LHS of 5b is

$$\begin{aligned}
& W_nW_{n+5}W_{n+6} - W_{n+3}W_{n+4}^2 \\
& = W_n\left((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\right) \\
& \qquad \times \left((p^5 - 4p^3q + 3pq^2)W_{n+1} - (p^4q - 3p^2q^2 + q^3)W_n\right) \\
& \qquad - \left((p^2 - q)W_{n+1} - pqW_n\right)\left((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\right)^2 \\
& = W_n^3(-pq^5 + 5p^3q^4 - 4p^5q^3 + p^7q^2) \\
& \quad W_n^2W_{n+1}(5p^2q^4 - 13p^4q^3 + 9p^6q^2 - 2p^8q) \\
& \quad W_nW_{n+1}^2(-pq^4 + p^3q^3 + 4p^5q^2 - 4p^7q + p^9) \\
& \quad W_{n+1}^3(4p^2q^3 - 8p^4q^2 + 5p^6q - p^8).
\end{aligned}$$

Since the LHS and RHS of identity 5b are equal, the identity is proved. □

We finally present the proof of 10.

Proof of 10. To prove 10, we require Cassini's identity

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

We also need the following identities

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n, \\ F_{n+3} &= 2F_{n+1} + F_n, \\ F_{n+4} &= 3F_{n+1} + 2F_n, \\ F_{n+6} &= 8F_{n+1} + 5F_n, \\ L_{n+3} &= 4F_{n+1} + 3F_n. \end{aligned}$$

These identities are obtained by the use of the recurrence for F_n and the fact that $L_{n+3} = F_{n+4} + F_{n+2}$. To prove 10, we write the LHS and RHS of identity 10 in terms of F_n and F_{n+1}. The RHS of 10 is

$$\begin{aligned} &(-1)^{n+1} L_{n+3}^2 \\ &= ((F_{n+2}F_n - F_{n+1}^2)L_{n+3}^2 \\ &= (F_n^2 + F_nF_{n+1} - F_{n+1}^2)(4F_{n+1} + 3F_n)^2 \\ &= 9F_n^4 + 33F_n^3F_{n+1} + 31F_n^2F_{n+1}^2 - 8F_nF_{n+1}^3 - 16F_{n+1}^4. \end{aligned}$$

The LHS of 10 is

$$\begin{aligned} &F_nF_{n+2}F_{n+4}F_{n+6} - F_{n+3}^4 \\ &= F_n(F_{n+1} + F_n)(3F_{n+1} + 2F_n)(8F_{n+1} + 5F_n) - (2F_{n+1} + F_n)^4 \\ &= 9F_n^4 + 33F_n^3F_{n+1} + 31F_n^2F_{n+1}^2 - 8F_nF_{n+1}^3 - 16F_{n+1}^4. \end{aligned}$$

Since the LHS and RHS of identity 10 are equal, the identity is proved. □

References

[1] S. Fairgrieve and H. W. Gould, *Product difference Fibonacci identities of Simson, Gelin-Ceaàro, Tagiuri, and generalizations*, The Fibonacci Quarterly **43.2** (2005), 137–141.
[2] V. E. Hoggatt Jr. and G. E. Bergum, *A problem of Fermat and the Fibonacci sequence*, The Fibonacci Quarterly **15.4** (1977), 323–330.
[3] A. F. Horadam, *Basic Properties of a Certain Generalized Sequence of Numbers*, The Fibonacci Quarterly **3.3** (1965), 161–176.
[4] R. S. Melham, *A Fibonacci identity in the spirit of Simson and Gelin-Cesàro*, The Fibonacci Quarterly **41.2** (2003), 142-143.

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