Proof of a Result by Jarden by Generalizing a Proof by Carlitz

Curtis Cooper and Robert E. Kennedy
Department of Mathematics, Central Missouri State University
Warrensburg, MO 64093

1. Introduction. Let $u_0 = 0$, $u_1 = 1$, and

$$u_n = au_{n-1} + bu_{n-2}$$

for any positive integer $n \geq 2$. Also, for any nonnegative integer m, define

$$\binom{m}{j}_u = \begin{cases} 1, & \text{if } j = 0; \\ \frac{u_m \cdots u_{m-j+1}}{u_j \cdots u_1}, & \text{if } j = 1, \ldots m. \end{cases}$$

In [1], Jarden showed that for any positive integer k,

$$\sum_{i=0}^{k+1} (-1)^{i(i+1)/2} b^{(i-1)i/2} \binom{k+1}{i}_u u_{n-i}^k = 0.$$

In this paper we will prove Jarden's result by generalizing a proof by Carlitz [2]. In addition, we will present a new like-power recurrence relation identity. Detailed proofs of the lemmas and the theorem will be supplied at the end of the paper.

2. Sequential Results. Let

$$\alpha, \beta = \frac{a \pm \sqrt{a^2 + 4b}}{2}.$$

<u>Lemma 2.1</u>. Let n be a non-negative integer. Then

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Lemma 2.2. Let $n \geq -1$ be an integer. Then

$$u_{n+1} = \sum_r \binom{r}{n-r} a^{2r-n} b^{n-r}.$$

Lemma 2.3. Let $n \geq 2$ be an integer. Then

(a) $u_n + bu_{n-2} = \alpha^{n-1} + \beta^{n-1}$.

(b) $bu_n u_{n-2} - bu_{n-1}^2 = \alpha^{n-1}\beta^{n-1}$.

Lemma 2.4. Let k be a positive integer and $0 \leq r \leq n$ be integers. Then

$$(u_k x + bu_{k-1})^r (u_{k+1}x + bu_k)^{n-r}$$

$$= \sum_{r_1,\ldots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} a^{kn-r-2r_1-\cdots-2r_{k-1}-r_k} b^{r_1+\cdots+r_k} x^{n-r_k}.$$

3. Matrix Results. Let

$$A_{n+1} = \left[\binom{r}{n-c} a^{r+c-n} b^{n-c}\right], \quad 0 \leq r, c \leq n,$$

be a matrix of order $n+1$. For example, for $n = 3$,

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & b & a \\ 0 & b^2 & 2ab & a^2 \\ b^3 & 3ab^2 & 3a^2b & a^3 \end{pmatrix}.$$

Lemma 3.1.

$$\mathrm{tr}(A_{n+1}^k) = \frac{u_{kn+k}}{u_k}$$

for any positive integer k.

It is worth noting that the case $k = 1$ is exactly Lemma 2.2 so that Lemma 3.1 is in some sense a generalization of Lemma 2.2.

Lemma 3.2. The eigenvalues of A_{n+1} are

$$\alpha^n, \alpha^{n-1}\beta, \ldots, \alpha\beta^{n-1}, \beta^n.$$

Lemma 3.3.

$$\prod_{j=0}^{n}(x-\alpha^j\beta^{n-j}) = \sum_{i=0}^{n+1}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{n+1}{i}_u x^{n+1-i}.$$

The next lemma is similar to a result in Hoggatt and Bicknell [3].

Lemma 3.4.

$$(A_{k+1}^n)_{k,i} = \binom{k}{i}u_{n+1}^i(bu_n)^{k-i}.$$

4. Jarden's Result.

Theorem 4.1.

$$\sum_{i=0}^{k+1}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{k+1}{i}_u u_{n-i}^k = 0.$$

5. More Results and Open Questions. More identities, like the one just derived, need to be studied. For example it can be shown, using the computer algebra system DERIVE, that if x_0, x_1, and x_2 are arbitrary and

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3},$$

then

$$x_n^2 = (a^2+b)x_{n-1}^2 + (a^2b+b^2+ac)x_{n-2}^2 + (a^3c+4abc-b^3+2c^2)x_{n-3}^2$$
$$(-ab^2c+a^2c^2-bc^2)x_{n-4}^2 + (b^2c^2-ac^3)x_{n-5}^2 - c^4x_{n-6}^2$$

Is there a similar formula for third powers? Also, what about formulas for

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} + dx_{n-4}?$$

6. Proofs.

Proof of Lemma 2.1. Let

$$G(z) = u_0 + u_1 z + u_2 z^2 + \cdots.$$

Then

$$azG(z) = au_0 z + au_1 z^2 + \cdots$$

and

$$bz^2 G(z) = bu_0 z^2 + \cdots.$$

Substracting the last two equations from the first and using the definition of u_n,

$$(1 - az - bz^2)G(z) = z$$

so

$$G(z) = \frac{z}{1 - az - bz^2} = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha z} - \frac{1}{1 - \beta z}\right).$$

Thus,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Proof of Lemma 2.2. By induction on n. First of all, the result is true for $n = -1$ and $n = 0$. Now assume that $n \geq 0$ and that the result is true for n and $n - 1$. Then

$$\begin{aligned}
u_{n+1} &= au_n + bu_{n-1} \\
&= a\sum_r \binom{r}{n-1-r} a^{2r-n+1} b^{n-1-r} + b\sum_r \binom{r}{n-2-r} a^{2r-n+2} b^{n-2-r} \\
&= \sum_r \left[\binom{r}{n-1-r} + \binom{r}{n-2-r}\right] a^{2r-n+2} b^{n-1-r} \\
&= \sum_r \binom{r+1}{n-1-r} a^{2r-n+2} b^{n-1-r} \\
&= \sum_r \binom{r}{n-r} a^{2r-n} b^{n-r}.
\end{aligned}$$

Proof of Lemma 2.3.

(a) By the definition of u_n and Lemma 2.1,

$$\begin{aligned} u_n + bu_{n-2} &= u_n + u_n - au_{n-1} \\ &= 2\frac{\alpha^n - \beta^n}{\alpha - \beta} - (\alpha+\beta)\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \\ &= \frac{2\alpha^n - 2\beta^n - \alpha^n + \alpha\beta^{n-1} - \beta\alpha^{n-1} + \beta^n}{\alpha - \beta} \\ &= \frac{\alpha^n - \beta^n + \alpha\beta^{n-1} - \beta\alpha^{n-1}}{\alpha - \beta} \\ &= \frac{\alpha(\alpha^{n-1} + \beta^{n-1}) - \beta(\alpha^{n-1} + \beta^{n-1})}{\alpha - \beta} \\ &= \frac{(\alpha - \beta)(\alpha^{n-1} + \beta^{n-1})}{\alpha - \beta} = \alpha^{n-1} + \beta^{n-1}. \end{aligned}$$

(b) By the definition of u_n and Lemma 2.1,

$$bu_n u_{n-2} - bu_{n-1}^2 = u_n(u_n - au_{n-1}) - bu_{n-1}^2$$

$$= u_n^2 - au_n u_{n-1} - bu_{n-1}^2$$

$$= u_n^2 - u_{n-1}(au_n + bu_{n-1}) = u_n^2 - u_{n-1}u_{n+1}$$

$$= \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{1}{(\alpha - \beta)^2}(\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n} - \alpha^{2n} - \beta^{2n} + \alpha^{n+1}\beta^{n-1} + \alpha^{n-1}\beta^{n+1})$$

$$= \frac{1}{(\alpha - \beta)^2}(\alpha^{n+1}\beta^{n-1} - 2\alpha^n\beta^n + \alpha^{n-1}\beta^{n+1})$$

$$= \frac{\alpha^{n-1}\beta^{n-1}}{(\alpha - \beta)^2}(\alpha^2 - 2\alpha\beta + \beta^2) = \alpha^{n-1}\beta^{n-1}.$$

Proof of Lemma 2.4. By induction on k. The result is true for $k = 1$, since

$$x^r(ax+b)^{n-r} = \sum_s \binom{n-r}{s} a^{n-r-s} b^s x^{n-s}.$$

Now assume the result is true for some positive integer k. In this result, substitute $a + bx^{-1}$ for x and multiply by x^n. The left side of this equation is

$$(au_k x + bu_{k-1}x + bu_k)^r (au_{k+1}x + bu_k x + bu_{k+1})^{n-r}$$

which is equal to

$$(u_{k+1}x + bu_k)^r (u_{k+2}x + bu_{k+1})^{n-r}.$$

Expanding the right side of this equation and simplifying, we obtain

$$\sum_{r_1,\ldots,r_{k+1}} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_k}{r_{k+1}} \cdot a^{(k+1)n-r-2r_1-\cdots-2r_k-r_{k+1}} b^{r_1+\cdots+r_{k+1}} x^{n-r_{k+1}}.$$

Therefore, the result is proved.

Proof of Lemma 3.1. We first recall Lemma 2.4, that is, for any positive integer k,

$$(u_k x + bu_{k-1})^r (u_{k+1}x + bu_k)^{n-r}$$
$$= \sum_{r_1,\ldots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} a^{kn-r-2r_1-\cdots-2r_{k-1}-r_k} b^{r_1+\cdots+r_k} x^{n-r_k}.$$

Multiplying both sides of this equation by x^r and summing over r, we have

$$\sum_{r=0}^{n} x^r (u_k x + bu_{k-1})^r (u_{k+1}x + bu_k)^{n-r}$$
$$= \sum_{r,r_1,\cdots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} \cdot a^{kn-r-2r_1-\cdots-2r_{k-1}-r_k} b^{r_1+\cdots r_k} x^{n+r-r_k}.$$

The coefficient of x^n on the right side of this equation is

$$\mathrm{tr}(A_{n+1}^k).$$

The coefficient of x^n on the left side of this equation is

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} u_k^s (bu_{k-1})^{r-s} u_{k+1}^t (bu_k)^{n-r-t}$$
$$= \sum_{r+s\le n} \binom{r}{s}\binom{n-r}{s} (bu_{k-1})^{r-s} u_k^s u_{k+1}^{n-r-s} (bu_k)^s.$$

Let v_n be this last term. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} v_n x^n &= \sum_{r,s=0}^{\infty} \binom{r}{s} b^r u_{k-1}^{r-s} u_k^{2s} x^{r+s} \sum_{n=r+s}^{\infty} \binom{n-r}{s} (u_{k+1}x)^{n-r-s} \\
&= \sum_{r,s=0}^{\infty} \binom{r}{s} b^r u_{k-1}^{r-s} u_k^{2s} x^{r+s} (1-u_{k+1}x)^{-s-1} \\
&= \sum_{s=0}^{\infty} b^s u_k^{2s} x^{2s} (1-u_{k+1}x)^{-s-1} \sum_{r\geq s} \binom{r}{s} (bu_{k-1}x)^{r-s} \\
&= \sum_{s=0}^{\infty} b^s u_k^{2s} x^{2s} (1-u_{k+1}x)^{-s-1} (1-bu_{k-1}x)^{-s-1} \\
&= \frac{1}{(1-u_{k+1}x)(1-bu_{k-1}x)} \frac{1}{1-\frac{bu_k^2x^2}{(1-u_{k+1}x)(1-bu_{k-1}x)}} \\
&= \frac{1}{(1-u_{k+1}x)(1-bu_{k-1}x)-bu_k^2x^2} \\
&= \frac{1}{1-(u_{k+1}+bu_{k-1})x+(bu_{k+1}u_{k-1}-bu_k^2)x^2}.
\end{aligned}$$

Next, by Lemma 2.3, the last expression is equal to

$$\frac{1}{1-(\alpha^k+\beta^k)x+\alpha^k\beta^k x^2} = \frac{1}{\alpha^k-\beta^k}\left(\frac{\alpha^k}{1-\alpha^k x} - \frac{\beta^k}{1-\beta^k x}\right).$$

Thus,

$$v_n = \frac{u_{kn+k}}{u_k}.$$

Therefore,

$$\mathrm{tr}(A_{n+1}^k) = \frac{u_{kn+k}}{u_k}.$$

Proof of Lemma 3.2. Let

$$f_{n+1}(x) = \det(xI - A_{n+1}).$$

If the eigenvalues of A_{n+1} are $\lambda_0, \lambda_1, \ldots, \lambda_n$, then by Lemmas 3.1 and 2.1,

$$\begin{aligned}
\frac{f'_{n+1}(x)}{f_{n+1}(x)} &= \sum_{j=0}^{n} \frac{1}{x-\lambda_j} = \sum_{k=0}^{\infty} x^{-k-1} \sum_{j=0}^{n} \lambda_j^k \\
&= \sum_{k=0}^{\infty} x^{-k-1} \operatorname{tr}(A_{n+1}^k) = \sum_{k=0}^{\infty} x^{-k-1} \frac{\alpha^{nk+k}-\beta^{nk+k}}{\alpha^k-\beta^k} \\
&= \sum_{k=0}^{\infty} x^{-k-1} \sum_{j=0}^{n} \alpha^{jk}\beta^{(n-j)k} = \sum_{j=0}^{n} \frac{1}{x-\alpha^j\beta^{n-j}}.
\end{aligned}$$

Thus,

$$f_{n+1}(x) = \prod_{j=0}^{n} (x-\alpha^j\beta^{n-j})$$

so the eigenvalues of A_{n+1} are

$$\alpha^n, \alpha^{n-1}\beta, \ldots, \alpha\beta^{n-1}, \beta^n.$$

Proof of Lemma 3.3. To begin the proof of Lemma 3.3, we need the identity

$$\prod_{j=0}^{n-1} (1-q^j x) = \sum_{i=0}^{n} (-1)^i q^{(i-1)i/2} \begin{bmatrix} n \\ i \end{bmatrix} x^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(1-q^n)\cdots(1-q^{n-i+1})}{(1-q^i)\cdots(1-q)}. \tag{1}$$

Replacing q in (1) by β/α and using Lemma 2.1, we find that $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$ is

$$\alpha^{i^2-ni} \binom{n}{i}_u.$$

Thus, (1) becomes

$$\prod_{j=0}^{n-1} (1-\alpha^{-j}\beta^j x) = \sum_{i=0}^{n} (-1)^i \alpha^{i(i+1)/2-ni} \beta^{(i-1)i/2} \binom{n}{i}_u x^i.$$

Substituting $\alpha^{n-1}x$ for x and using the fact that $\alpha\beta = -b$ we have

$$\begin{aligned}\prod_{j=0}^{n-1}(1-\alpha^{n-j-1}\beta^j x) &= \sum_{i=0}^{n}(-1)^i(\alpha\beta)^{(i-1)i/2}\binom{n}{i}_u x^i \\ &= \sum_{i=0}^{n}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{n}{i}_u x^i.\end{aligned}$$

Replacing x by x^{-1} gives

$$\prod_{j=0}^{n-1}(x-\alpha^{n-j-1}\beta^j) = \sum_{i=0}^{n}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{n}{i}_u x^{n-i},$$

which is what we wanted to prove.

Proof of Lemma 3.4. Let k be a fixed non-negative integer. We will prove the result by induction on n. The above equality is true for $n = 0$. Now assume the result is true for some $n \geq 0$. Then since $A_{k+1}^{n+1} = A_{k+1}^n \cdot A_{k+1}$,

$$\begin{aligned}(A_{k+1}^{n+1})_{k,i} &= \sum_{j=0}^{k}(A_{k+1}^n)_{k,j}(A_{k+1})_{j,i} \\ &= \sum_{j=0}^{k}\binom{k}{j}u_{n+1}^j(bu_n)^{k-j}\binom{j}{k-i}a^{j+i-k}b^{k-i}.\end{aligned}$$

To continue the equalities, we use the identity

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

to obtain

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{k}{k-i}\binom{i}{i+j-k}(bu_{n+1})^{k-i}(au_{n+1})^{i+j-k}(bu_n)^{k-j}\\
&=\binom{k}{i}(bu_{n+1})^{k-i}\sum_{j=0}^{k}\binom{i}{i+j-k}(au_{n+1})^{i+j-k}(bu_n)^{k-j}\\
&=\binom{k}{i}(bu_{n+1})^{k-i}\sum_{m=0}^{i}\binom{i}{m}(au_{n+1})^{m}(bu_n)^{i-m}\\
&=\binom{k}{i}(bu_{n+1})^{k-i}(au_{n+1}+bu_n)^{i}\\
&=\binom{k}{i}u_{n+2}^{i}(bu_{n+1})^{k-i}.
\end{aligned}$$

Thus, the result is true by induction on n.

Proof of Theorem 4.1. By Lemma 3.3, the characteristic polynomial of A_{k+1} is

$$\sum_{i=0}^{k+1}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{k+1}{i}_u x^{k+1-i}.$$

But, by the Cayley-Hamilton Theorem, every matrix satisfies its characteristic polynomial. Thus, for $n-1\geq k+1$,

$$\sum_{i=0}^{k+1}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{k+1}{i}_u A_{k+1}^{n-1-i}=O, \tag{2}$$

where O denotes the $(k+1)\times(k+1)$ zero matrix. Now, taking the result of Lemma 3.4 (with $i=k$ and $n=n-1-i$) and substituting this result into (2), we obtain Jarden's result.

References

1. D. Jarden, *Recurring Sequences,* 2nd ed., Riveon Lematika, (1966), Jerusalem, Israel, 30–33.

2. L. Carlitz, "The Characteristic Polynomial of a Certain Matrix of Binomial Coefficients," *The Fibonacci Quarterly,* 3 (1965), 81–89.

3. V. Hoggatt, Jr. and M. Bicknell, "Fourth Power Fibonacci Identities from Pascal's Triangle," *The Fibonacci Quarterly,* 2 (1964), 261–266.

AMS numbers: 11B37, 11B65