ON THE NATURAL DENSITY OF THE k-ZECKENDORF NIVEN NUMBERS

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ABSTRACT. The k-generalized Fibonacci numbers are defined as

$$F_n^{(k)} = \begin{cases} 0, & \text{for } n < k-1 \\ 1, & \text{for } n = k-1 \\ \sum_{i=1}^{k} F_{n-i}^{(k)}, & \text{for } n \geq k. \end{cases}$$

The k-Zeckendorf representation of a positive integer is defined as

$$n = \sum_{i \geq k} \epsilon_i F_i^{(k)},$$

where $\epsilon_i \in \{0,1\}$ and for all $i \geq k$ we have $\epsilon_i \epsilon_{i+1} \cdots \epsilon_{i+k-1} = 0$. Given this representation of a number n we say the k-Zeckendorf digital sum of n is $z_k(n) = \sum_{i \geq k} \epsilon_i$ and if $z_k(n) \big| n$ then n is called a k-Zeckendorf Niven number. We prove that the natural density of the k-Zeckendorf Niven numbers is zero.

1. Introduction

Niven numbers are a well known topic in number theory. A Niven number is a number which is divisible by the sum of its digits (its digital sum). For example 12 is a Niven number since the sum of its digits is 3, and 3 divides 12. But 25 is not Niven because the sum of its digits is 7, which does not divide 25. Niven numbers are named after Ivan Niven, the mathematician who motivated the study of these numbers in 1977.

The Zeckendorf representation of a number is another well known topic in number theory. We obtain the Zeckendorf representation of a positive number by representing it as the sum of non consecutive Fibonacci numbers. It is well known that this representation is unique and can be obtained by using the greedy algorithm. For example the Zeckendorf representation of 100 is $89+8+3$. The Zeckendorf representation is attributed to the number theorist Edouard Zeckendorf who among other things was an officer in the Belgian Army and a medical doctor.

2. Definitions

We begin by defining the k-generalized Fibonacci numbers which are a generalization of the normal Fibonacci numbers, which are obtained when $k = 2$.

Definition 2.1. *Let $k \geq 2$ be an integer. The k-generalized Fibonacci numbers are defined by*

$$F_n^{(k)} = \begin{cases} 0, & \text{for } n < k-1 \\ 1, & \text{for } n = k-1 \\ \sum_{i=1}^{k} F_{n-i}^{(k)}, & \text{for } n \geq k. \end{cases}$$

For example the 3-generalized Fibonacci numbers for $n \geq 0$ would be

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots$$

We will now define the k-Zeckendorf representation of a non-negative integer. This is a generalization of the well known Zeckendorf representation of a number, which we obtain when $k = 2$.

Definition 2.2. *Let $k \geq 2$ be an integer. The k-Zeckendorf representation of a nonnegative integer n is the unique representation of n as*

$$n = \sum_{i \geq k} \epsilon_i F_i^{(k)}$$

where $\epsilon_i \in \{0, 1\}$ and for all $i \geq k$ we have $\epsilon_i \epsilon_{i+1} \cdots \epsilon_{i+k-1} = 0$.

It can easily be seen that every integer has at least one representation of this form by applying the greedy algorithm. However, to show this representation is unique, we can apply a Theorem proved by Fraenkel that deals with unique representation in a numeration system defined by a sequence [2, Theorem 2].

The shorthand we will use to write a number in its k-Zeckendorf representation, will be to treat ϵ_k as the least significant digit, ϵ_{k+1} as the next most significant digit, and so on. For example the 3-Zeckendorf representation of 12 is 1101 since $12 = F_3^{(3)} + F_5^{(3)} + F_6^{(3)}$.

This representation would be the natural way to represent a number with 1's and 0's using no more than $k-1$ 1's in a row.

Definition 2.3. *Let n be a nonnegative integer, $k \geq 2$ be an integer, and $n = \sum_{i \geq k} \epsilon_i F_i^{(k)}$ be the k-Zeckendorf representation of n. Then the k-Zeckendorf digital sum is defined to be*

$$z_k(n) = \sum_{i \geq k} \epsilon_i.$$

For example $z_3(12) = 3$ since $12 = F_3^{(3)} + F_5^{(3)} + F_6^{(3)}$.

A Niven number is one that is divisible by the sum of its digits. We generalize this definition to the k-Zeckendorf representation as follows.

Definition 2.4. *Let n be a positive integer, and $k \geq 2$ be an integer. Then n is k-Zeckendorf Niven number if and only if $z_k(n) \big| n$.*

Thus our example, 12 is 3-Zeckendorf Niven since $z_3(12) \big| 12$.

3. Natural Density

We begin by defining natural density.

Definition 3.1. *If A is a set of positive integers and $A(x) = \#\{a \in A : a \leq x\}$, where $\#$ means the number of elements in the set, then the "natural density" of A is defined as*

$$\lim_{x\to\infty} \frac{A(x)}{x},$$

if the limit exists.

The natural density is the portion of the natural numbers included in some set. It is one way of measuring the size of subsets of the natural numbers.

We will prove that the natural density of the k-Zeckendorf Niven numbers is zero. To do this we will use a theorem proved by Cooper and Kennedy [1, Theorem 2].

Theorem 3.1. *Let a_n be the nth term of an increasing sequence and for each n, let μ_n = mean $f([0, a_n))$ and σ_n^2 = variance $f([0, a_n))$ for an integer-valued function f. Then for $S = \{n : f(n)|n\}$ and $S(x) = \#\{n \in S : n \leq x\}$, if*

$$\lim_{n\to\infty} \mu_n = \infty,$$
$$\lim_{n\to\infty} \frac{\mu_n}{\sigma_n} = \infty,$$

and if a_n/a_{n-1} is bounded, then

$$\lim_{n\to\infty} \frac{S(x)}{x} = 0.$$

This theorem was used in Winter's thesis to prove that the natural density of the base b Niven numbers is zero [5, Theorem 2.1]. Base b Niven numbers are those who are divisible by the sum of their digits in the base b representation of the number.

Cooper and Kennedy also proved that the natural density of the 2-Zeckendorf Niven numbers is zero [3]. That is the base case where a number is divisible by the number of Fibonacci numbers in its Zeckendorf representation.

4. The φ_k function

To find the natural density of the k-Zeckendorf Niven numbers using Theorem 3.1, we need to determine the mean and variance of the k-Zeckendorf digital sums. To find these we must first determine

$$\sum_{0\leq n<x} z_k(n) \text{ and } \sum_{0\leq n<x} (z_k(n))^2.$$

In an effort to find these formulas we will make the following definition.

Definition 4.1. *Let n and m be integers; we define*

$$\varphi_k(n, m) = \#\{0 \leq i < F_n^{(k)} : z_k(i) = m\}.$$

The φ_k function counts the number of nonnegative integers less than $F_n^{(k)}$ which have a k-Zeckendorf digital sum of m. Note that the k-Zeckendorf representation of $F_n^{(k)}$ ($n \geq k$) is a 1 followed by $n - k$ 0's.

Lemma 4.1. *For $n \geq k$ we have*

$$\varphi_k(n,m) = \sum_{i=0}^{k-1} \varphi_k(n-i-1, m-i).$$

Proof. We will obtain a recursive formula for $\varphi_k(n,m)$ by partitioning the numbers from 0 to $F_n^{(k)} - 1$ whose k-Zeckendorf digital sum is m into k disjoint sets. To partition the numbers whose k-Zeckendorf digital sum is m, we look at the k-Zeckendorf digital expansion of the numbers from 0 to $F_n^{(k)} - 1$, i.e., the digits

$$\epsilon_{n-1}\epsilon_{n-2}\cdots\epsilon_k.$$

In our partition, the first set consists of the numbers with k-Zeckendorf digital sum m whose k-Zeckendorf representation has $\epsilon_{n-1} = 0$. The number of elements in this set is $\varphi_k(n-1,m)$. The next set, disjoint from the first set, is the set consisting of the numbers with k-Zeckendorf digital sum m whose k-Zeckendorf representation has $\epsilon_{n-1} = 1$ and $\epsilon_{n-2} = 0$. The number of elements in this set is $\varphi_k(n-2, m-1)$. The third set will consist of the numbers with k-Zeckendorf digital sum m whose k-Zeckendorf representation is $\epsilon_{n-1} = 1$, $\epsilon_{n-2} = 1$, and $\epsilon_{n-3} = 0$. This set is disjoint from the first and second sets we have given. And the number of elements in this set is $\varphi_k(n-3, m-2)$. Continue this process until we get to the kth set consisting of the numbers with k-Zeckendorf digital sum m whose k-Zeckendorf representation is $\epsilon_{n-1} = 1$, $\epsilon_{n-2} = 1$, …, $\epsilon_{n-k+1} = 1$, and $\epsilon_{n-k} = 0$. This set is disjoint from all the other sets. And the number of elements in this set is $\varphi_k(n-k, m-k+1)$. And all of these sets unioned together give us all the numbers up to $F_n^{(k)}$ with a k-Zeckendorf digital sum of m. Therefore, the recurrence relation has just been proved. □

Remark 4.1. *Note that* $\varphi_k(n,m) = 0$ *if* $m > n-k \geq 0$ *or* $m < 0$ *or* $n < k-1$.

Remark 4.2. *Note that* $\varphi_k(n,0) = 1$ *if* $n \geq k-1$.

Remark 4.3. *Note that*

$$\sum_{m\in\mathbb{Z}} \varphi_k(n,m) = F_n^{(k)}.$$

5. The $M_t^{(k)}$ function

Definition 5.1. *Let t be a nonnegative integer. Then*

$$M_t^{(k)}(n) = \sum_{m\in\mathbb{Z}} m^t \varphi_k(n,m).$$

Remark 5.1. *Note that*

$$M_1^{(k)}(n) = \sum_{0\leq i<F_n^{(k)}} z_k(i)$$

and

$$M_2^{(k)}(n) = \sum_{0\leq i<F_n^{(k)}} \left(z_k(i)\right)^2.$$

Unfortunately, our current definitions of $M_1^{(k)}$ and $M_2^{(k)}$ are not very easy to compute. Therefore, we will apply what we know about φ_k by substituting our recursive definition of φ_k into the definition of $M_1^{(k)}(n)$ and $M_2^{(k)}(n)$. Doing so we arrive at the following recursive definition of $M_1^{(k)}(n)$.

Theorem 5.2. *Let $k \geq 2$ and $n \geq k$ be integers. Then*

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1} \left(M_1^{(k)}(n-i-1) + iF_{n-i-1}^{(k)} \right).$$

Proof. By the definition of $M_1^{(k)}$,

$$M_1^{(k)}(n) = \sum_{m \in \mathbb{Z}} m\varphi_k(n,m).$$

Substituting for the recursive definition of φ_k and distributing m, we obtain

$$M_1^{(k)}(n) = \sum_{m \in \mathbb{Z}} \left(\sum_{i=0}^{k-1} m\varphi_k(n-i-1, m-i) \right).$$

Switching the order of the sums and rewriting m, we obtain

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} (m-i+i)\varphi_k(n-i-1, m-i) \right).$$

Reindexing the inner sums, we get

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} (m+i)\varphi_k(n-i-1, m) \right).$$

Upon distributing φ_k we obtain

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} m\varphi_k(n-i-1, m) + \sum_{m \in \mathbb{Z}} i\varphi_k(n-i-1, m) \right).$$

Substituting from the definition of $M_1^{(k)}$ and Remark 4.3, we get

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1} \left(M_1^{(k)}(n-i-1) + iF_{n-i-1}^{(k)} \right).$$

□

We get a similar result for $M_2^{(k)}(n)$.

Theorem 5.3. *Let $k \geq 2$ and $n \geq k$ be integers. Then*

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1} \left(M_2^{(k)}(n-i-1) + 2iM_1^{(k)}(n-i-1) + i^2 F_{n-i-1}^{(k)} \right).$$

Proof. By the definition of $M_2^{(k)}$, we know that

$$M_2^{(k)}(n) = \sum_{m \in \mathbb{Z}} m^2 \varphi_k(n,m).$$

Substituting the recursive definition of φ_k and distributing m^2, we obtain

$$M_2^{(k)}(n) = \sum_{m \in \mathbb{Z}} \left(\sum_{i=0}^{k-1} m^2 \varphi_k(n-i-1, m-i) \right).$$

Switching the order of the sums we obtain

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} m^2 \varphi_k(n-i-1, m-i) \right).$$

By reindexing the inner sums, we get

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} (m+i)^2 \varphi_k(n-i-1, m) \right).$$

Then, we rewrite $(m+i)^2$ as $m^2 + 2im + i^2$ to get

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} (m^2 + 2im + i^2) \varphi_k(n-i-1, m) \right).$$

Thus, by distributing φ_k and splitting up the inner sums, we obtain

$$\begin{aligned} M_2^{(k)}(n) &= \sum_{i=0}^{k-1} \left(\sum_{m \in \mathbb{Z}} m^2 \varphi_k(n-i-1, m) + \sum_{m \in \mathbb{Z}} 2im \varphi_k(n-i-1, m) \right. \\ &+ \left. \sum_{m \in \mathbb{Z}} i^2 \varphi_k(n-i-1, m) \right). \end{aligned}$$

Substituting for the definition of $M_1^{(k)}$, $M_2^{(k)}$, and Remark 4.3, we get

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1} \left(M_2^{(k)}(n-i-1) + 2i M_1^{(k)}(n-i-1) + i^2 F_{n-i-1}^{(k)} \right).$$

□

6. Closed form for $F_n^{(k)}$, $M_1^{(k)}$, and $M_2^{(k)}$

We wish to find a closed form for $F_n^{(k)}$, $M_1^{(k)}(n)$, and $M_2^{(k)}(n)$, or at least an approximation. This will make it easier to apply Cooper and Kennedy's Theorem 3.1 to prove the natural density of the k-Zeckendorf Niven numbers is zero. But before we can do that we need to prove the following lemmas that will help us with simplifying the closed form.

Lemma 6.1. *Let* $r_k(z) = 1 - \sum_{i=1}^{k} z^i$. *Then* $r_k(z) = \prod_{i=0}^{k-1} (1 - \alpha_i z)$ *where* α_0 *is real with* $1 < \alpha_0 < 2$ *and all the remaining* α_i*'s lie within the unit circle of the complex plane. Furthermore, all of the* α_i*'s are distinct.*

Proof. Let $f_k(z) = z^k - z^{k-1} - \cdots - z - 1$. Miller [4] proved that f_k has a real zero α_0 such that $1 < \alpha_0 < 2$, the remaining $k-1$ zeros of f_k, $\{\alpha_i : i = 1, \ldots, k-1\}$ lie within the unit circle in the complex plane, and the zeros of f_k are all simple (i.e., distinct.) Thus $r_k(z) = z^k f_k(\frac{1}{z}) = z^k \prod_{i=0}^{k-1}(\frac{1}{z} - \alpha_i) = \prod_{i=0}^{k-1}(1 - \alpha_i z)$. □

Remark 6.1. *The derivative of $r_k(z)$ is*

$$r_k'(z) = -\sum_{i=1}^{k} i z^{i-1} = \sum_{i=0}^{k-1}\left[-\alpha_i \prod_{\substack{j=0\\ j\neq i}}^{k-1}(1-\alpha_j z)\right]$$

and

$$r_k'\left(\frac{1}{\alpha_i}\right) = -\sum_{m=1}^{k} m\left(\frac{1}{\alpha_i}\right)^{m-1} = -\alpha_i \prod_{\substack{j=0\\ j\neq i}}^{k-1}\left(1-\alpha_j\left(\frac{1}{\alpha_i}\right)\right)$$

Lemma 6.2. *Let $r_k(z) = \prod_{i=0}^{k-1}(1-\alpha_i z)$ where α_i, $i = 0, 1, \ldots, k-1$ are the roots of f_k given in the proof of Lemma 6.1. Then*

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[2 - (k+1)\left(\frac{1}{\alpha_i}\right)^k\right]^p$$

Proof. We begin by simply combining the factors to obtain

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[\left(\frac{1}{\alpha_i} - 1\right) r_k'\left(\frac{1}{\alpha_i}\right)\right]^p.$$

Then we substitute the result of Remark 6.1 and multiply it out to obtain

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[-\sum_{m=1}^{k} m\left(\frac{1}{\alpha_i}\right)^m + \sum_{m=1}^{k} m\left(\frac{1}{\alpha_i}\right)^{m-1}\right]^p.$$

Combining terms, we obtain

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[1 + \sum_{m=1}^{k-1}\left(\frac{1}{\alpha_i}\right)^m - k\left(\frac{1}{\alpha_i}\right)^k\right]^p.$$

This can be rewritten as

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[2 - \left(1 - \sum_{m=1}^{k}\left(\frac{1}{\alpha_i}\right)^m\right) - (k+1)\left(\frac{1}{\alpha_i}\right)^k\right]^p.$$

However, $r\left(\frac{1}{\alpha_i}\right) = 1 - \sum_{m=1}^{k}\left(\frac{1}{\alpha_i}\right)^m$ is zero. Thus, we have

$$\left(\frac{1}{\alpha_i} - 1\right)^p \left[r_k'\left(\frac{1}{\alpha_i}\right)\right]^p = \left[2 - (k+1)\left(\frac{1}{\alpha_i}\right)^k\right]^p.$$

□

Lemma 6.3. *Let α_i, $i = 0, 1, \ldots, k-1$ be the roots of f_k given in the proof of Lemma 6.1. Then*

$$\left(\frac{1}{\alpha_0} - 1\right)^p \left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p = \left[-2\left(\frac{1}{\alpha_0}\right)^2 + \left(\frac{1}{\alpha_0}\right)^{k+2} + k\left(\frac{1}{\alpha_0}\right)^{k+1}\right]^p.$$

Proof. We begin by factoring out a common factor and reindexing the sum to obtain

$$\left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p = \left[\left(\frac{1}{\alpha_0}\right)^2 \sum_{m=1}^{k-1} m \left(\frac{1}{\alpha_0}\right)^{m-1}\right]^p.$$

Using Remark 6.1 we rewrite this as

$$\left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p = \left[\left(\frac{1}{\alpha_0}\right)^2 \left(-r_k'\left(\frac{1}{\alpha_0}\right) - k\left(\frac{1}{\alpha_0}\right)^{k-1}\right)\right]^p.$$

Distributing $\left(\frac{1}{\alpha_0}\right)^2$ and replacing $r_k'\left(\frac{1}{\alpha_0}\right)$ with the result from Lemma 6.2 we get

$$\left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p = \left[-\left(\frac{1}{\alpha_0}\right)^2 \frac{\left[2-(k+1)\left(\frac{1}{\alpha_i}\right)^k\right]}{\left(\frac{1}{\alpha_i}-1\right)} - k\left(\frac{1}{\alpha_0}\right)^{k+1}\right]^p.$$

Multiplying each side of this equation by $\left(\frac{1}{\alpha_0} - 1\right)^p$ we obtain

$$\begin{aligned}&\left(\frac{1}{\alpha_0} - 1\right)^p \left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p \\ = \ & \left[-\left(\frac{1}{\alpha_0}\right)^2 \left[2-(k+1)\left(\frac{1}{\alpha_i}\right)^k\right] - k\left(\frac{1}{\alpha_0}-1\right)\left(\frac{1}{\alpha_0}\right)^{k+1}\right]^p.\end{aligned}$$

Simplifying the right hand side we obtain

$$\left(\frac{1}{\alpha_0} - 1\right)^p \left[\sum_{i=1}^{k} (i-1) \left(\frac{1}{\alpha_0}\right)^i\right]^p = \left[-2\left(\frac{1}{\alpha_0}\right)^2 + \left(\frac{1}{\alpha_0}\right)^{k+2} + k\left(\frac{1}{\alpha_0}\right)^{k+1}\right]^p.$$

□

Using generating function techniques and the previous lemmas we can derive the following closed formula for the k-generalized Fibonacci numbers.

Lemma 6.4. *Let $n \geq 0$ and $k \geq 2$ be integers. Then a closed form of the k-generalized Fibonacci numbers is*

$$F_n^{(k)} = \sum_{i=0}^{k-1} \frac{\alpha_i^2 - \alpha_i}{2\alpha_i^k - (k+1)} \alpha_i^n,$$

where the α_i's are given in Lemma 6.1.

Proof. Define

$$F(z) = \sum_{n\geq 0} z^n F_n^{(k)}.$$

Then, substituting the recursive definition of $F_n^{(k)}$ we get

$$F(z) = z^{k-1} + \sum_{n \geq k} \left(z^n \sum_{i=1}^{k} F_{n-i}^{(k)} \right).$$

Distributing the z^n we obtain

$$F(z) = z^{k-1} + \sum_{n \geq k} \left(\sum_{i=1}^{k} z^n F_{n-i}^{(k)} \right).$$

Switching the order of the sums and factoring out z^i we get

$$F(z) = z^{k-1} + \sum_{i=1}^{k} \left(z^i \sum_{n \geq k} z^{n-i} F_{n-i}^{(k)} \right).$$

Now we reindex the inner sum to get

$$F(z) = z^{k-1} + \sum_{i=1}^{k} \left(z^i \sum_{n \geq k-i} z^n F_n^{(k)} \right).$$

But since $F_n^{(k)} = 0$ for $n < k-1$, we can add these terms to the sum to produce

$$F(z) = z^{k-1} + \sum_{i=1}^{k} \left(z^i \sum_{n \geq 0} z^n F_n^{(k)} \right).$$

Thus, substituting for $F(z)$ we have

$$F(z) = z^{k-1} + \sum_{i=1}^{k} z^i F(z).$$

Solving for $F(z)$ we obtain

$$F(z) = \frac{z^{k-1}}{1 - \sum_{i=1}^{k} z^i}.$$

Therefore by Lemma 6.1, we have

$$F(z) = \frac{z^{k-1}}{\prod_{i=0}^{k-1} (1 - \alpha_i z)}.$$

Using the partial fraction decomposition

$$\sum_{i=0}^{k-1} \frac{A_i}{1 - \alpha_i z}$$

and geometric series, this is equal to

$$\sum_{i=0}^{k-1} \left(A_i \sum_{n \geq 0} (\alpha_i z)^n \right).$$

Therefore,

$$F_n^{(k)} = \sum_{i=0}^{k-1} A_i \alpha_i^n.$$

Thus, we can solve for A_i. To do this, we begin with

$$\frac{z^{k-1}}{\prod_{i=0}^{k-1}(1-\alpha_i z)} = \sum_{i=0}^{k-1} \frac{A_i}{1-\alpha_i z}.$$

We then multiply each side by $\prod_{i=0}^{k-1}(1-\alpha_i z)$ to get

$$z^{k-1} = \sum_{i=0}^{k-1} \left[A_i \prod_{\substack{j=0 \\ j\neq i}}^{k-1} (1-\alpha_j z) \right].$$

We can then substitute $z = \frac{1}{\alpha_i}$. Note that this makes all terms in the sum zero except for the i^{th} term, and so

$$\left(\frac{1}{\alpha_i}\right)^{k-1} = A_i \prod_{\substack{j=0 \\ j\neq i}}^{k-1} \left(1-\alpha_j \frac{1}{\alpha_i}\right).$$

Therefore,

$$A_i = \frac{\left(\frac{1}{\alpha_i}\right)^{k-1}}{\prod_{\substack{j=0 \\ j\neq i}}^{k-1} \left(1-\alpha_j \frac{1}{\alpha_i}\right)}.$$

But by Remark 6.1, this is

$$A_i = \frac{-\left(\frac{1}{\alpha_i}\right)^{k-2}}{r_k'\left(\frac{1}{\alpha_i}\right)}.$$

Now we apply the result of Lemma 6.2 to get

$$A_i = \frac{-\left(\frac{1}{\alpha_i}\right)^{k-2}\left(\frac{1}{\alpha_i}-1\right)}{2-(k+1)\left(\frac{1}{\alpha_i}\right)^{k}}.$$

Distributing the top and multiplying the top and bottom by α_i^k we get

$$A_i = \frac{\alpha_i^2-\alpha_i}{2\alpha_i^k-(k+1)}.$$

Therefore, if we substitute this result back into the earlier equation we obtain

$$F_n^{(k)} = \sum_{i=0}^{k-1} \frac{\alpha_i^2-\alpha_i}{2\alpha_i^k-(k+1)}\alpha_i^n.$$

□

In a similar manner we can find an approximate formula for $M_1^{(k)}(n)$.

Theorem 6.2. *Let $k \geq 2$ and $n \geq k$ be integers. Let α_0 be as defined in Lemma 6.1. Then*

$$M_1^{(k)}(n) = \frac{(\alpha_0 - 1)\left(2\alpha_0^k - 1 - k\alpha_0\right)}{\left[2\alpha_0^k - (k+1)\right]^2} n\alpha_0^n + O(\alpha_0^n).$$

Proof. We begin with the definition of $M_1^{(k)}$,

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1}\left(M_1^{(k)}(n-i-1) + iF_{n-i-1}^{(k)}\right),$$

that we will reindex to get

$$M_1^{(k)}(n) = \sum_{i=1}^{k}\left(M_1^{(k)}(n-i) + (i-1)F_{n-i}^{(k)}\right).$$

Define the generating function $G_1(z)$ as

$$G_1(z) = \sum_{n\geq 0} M_1^{(k)}(n)z^n.$$

Substituting the previous recursive definition of $M_1^{(k)}(n)$ we obtain

$$G_1(z) = \sum_{n\geq 0}\left[\sum_{i=1}^{k}\left(M_1^{(k)}(n-i) + (i-1)F_{n-i}^{(k)}\right)\right] z^n.$$

Distributing the z^n, splitting up the sums, reordering them, and factoring $i-1$ out of the left inner sum we obtain

$$G_1(z) = \sum_{i=1}^{k}\sum_{n\geq 0} M_1^{(k)}(n-i)z^n + \sum_{i=1}^{k}\left[(i-1)\sum_{n\geq 0} F_{n-i}^{(k)} z^n\right].$$

Factoring z^i out of each inner sum and reindexing them we obtain

$$G_1(z) = \sum_{i=1}^{k}\left[z^i \sum_{n\geq -i} M_1^{(k)}(n)z^n\right] + \sum_{i=1}^{k}\left[(i-1)z^i \sum_{n\geq -i} F_n^{(k)} z^n\right].$$

But since $M_1^{(k)}(n)$ and $F_n^{(k)}$ are zero for $n < 0$, we can delete these terms to get

$$G_1(z) = \sum_{i=1}^{k}\left[z^i \sum_{n\geq 0} M_1^{(k)}(n)z^n\right] + \sum_{i=1}^{k}\left[(i-1)z^i \sum_{n\geq 0} F_n^{(k)} z^n\right].$$

Now we can substitute $G_1(z)$ for $\sum_{n\geq 0} M_1^{(k)}(n)z^n$ and $F(z)$ for $\sum_{n\geq 0} F_n^{(k)} z^n$ to get

$$G_1(z) = \sum_{i=1}^{k} z^i G_1(z) + \sum_{i=1}^{k}(i-1)z^i F(z).$$

Solving for $G_1(z)$ we obtain

$$G_1(z) = \frac{F(z)\sum_{i=1}^{k}(i-1)z^i}{1 - \sum_{i=1}^{k} z^i}$$

To complete the generating function $G_1(z)$ we substitute $F(z) = \frac{z^{k-1}}{1-\sum_{i=1}^{k} z^i}$ to get

$$G_1(z) = \frac{\frac{z^{k-1}}{1-\sum_{i=1}^{k} z^i} \sum_{i=1}^{k}(i-1)z^i}{1-\sum_{i=1}^{k} z^i}.$$

This simplifies to

$$G_1(z) = \frac{z^{k-1}\sum_{i=1}^{k}(i-1)z^i}{\left(1-\sum_{i=1}^{k} z^i\right)^2}.$$

Using the factorization of $1 - \sum_{i=1}^{k} z^i$ given in Lemma 6.1, we can rewrite the right hand side to get

$$G_1(z) = \frac{z^{k-1}\sum_{i=1}^{k}(i-1)z^i}{\prod_{i=0}^{k-1}(1-\alpha_i z)^2}.$$

Using partial fractions we have

$$\sum_{i=0}^{k-1}\left(\frac{A_i}{1-\alpha_i z} + \frac{B_i}{(1-\alpha_i z)^2}\right),$$

and then by geometric series, this is equal to

$$\sum_{i=0}^{k-1}\left(A_i\sum_{n\geq 0}(\alpha_i z)^n + B_i\sum_{n\geq 0}(n+1)(\alpha_i z)^n\right).$$

Since we defined $G_1(z) = \sum_{n\geq 0} M_1^{(k)}(n)z^n$ we have

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1}\left(A_i\alpha_i^n + B_i(n+1)\alpha_i^n\right).$$

This becomes

$$M_1^{(k)}(n) = \sum_{i=0}^{k-1}\left((A_i + B_i(n+1))\alpha_i^n\right).$$

Now the only term of significance as $n \to \infty$ on the right hand of this equation is the term with α_0 since $1 < \alpha_0 < 2$ and $\|\alpha_i\| < 1$ for all $i \neq 0$. Of that term only the constant B_0 matters as $n \to \infty$ since it is multiplied by $(n+1)$ and A_0 is just a constant. Therefore, we can restate the equation using the big O notation and Miller's facts about the α_i's.

$$M_1^{(k)}(n) = B_0 n\alpha_0^n + O(\alpha_0^n).$$

Thus, we must solve for B_0. To do that we will begin with

$$\frac{z^{k-1}\sum_{i=1}^{k}(i-1)z^i}{\prod_{i=0}^{k-1}(1-\alpha_i z)^2} = \sum_{i=0}^{k-1}\left(\frac{A_i}{1-\alpha_i z} + \frac{B_i}{(1-\alpha_i z)^2}\right).$$

We then multiply each side by $\prod_{i=0}^{k-1}(1-\alpha_i z)^2$ to get

$$z^{k-1}\sum_{i=1}^{k}(i-1)z^i = \sum_{i=0}^{k-1}\left[(A_i(1-\alpha_i z)+B_i)\prod_{\substack{j=0\\ j\neq i}}^{k-1}(1-\alpha_j z)^2\right].$$

We can then substitute $z=\frac{1}{\alpha_0}$. Note that this makes all terms in the right sum zero except for the first term ($i=0$) and this yields

$$\left(\frac{1}{\alpha_0}\right)^{k-1}\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^i = \left[A_0\left(1-\alpha_0\frac{1}{\alpha_0}\right)+B_0\right]\prod_{\substack{j=0\\ j\neq 0}}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^2.$$

This simplifies to

$$\left(\frac{1}{\alpha_0}\right)^{k-1}\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^i = B_0\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^2.$$

Therefore,

$$B_0 = \frac{\left(\frac{1}{\alpha_0}\right)^{k-1}\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^i}{\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^2}.$$

Multiply the numerator and denominator by $\left(\frac{1}{\alpha_0}-1\right)^2\alpha_0^2$ to obtain

$$B_0 = \frac{\left(\frac{1}{\alpha_0}\right)^{k-3}\left(\frac{1}{\alpha_0}-1\right)^2\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^i}{\left(\frac{1}{\alpha_0}-1\right)^2\alpha_0^2\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^2}.$$

By applying Lemma 6.3 we obtain

$$B_0 = \frac{\left(\frac{1}{\alpha_0}\right)^{k-3}\left(\frac{1}{\alpha_0}-1\right)\left[-2\left(\frac{1}{\alpha_0}\right)^2+\left(\frac{1}{\alpha_0}\right)^{k+2}+k\left(\frac{1}{\alpha_0}\right)^{k+1}\right]}{\left[2-(k+1)\left(\frac{1}{\alpha_i}\right)^k\right]^2}.$$

Simplifying and multiplying the numerator and denominator by α_0^{2k} we get

$$B_0 = \frac{(\alpha_0-1)\left(2\alpha_0^k-1-k\alpha_0\right)}{\left[2\alpha_0^k-(k+1)\right]^2}.$$

Therefore, if we substitute this result back into the earlier equation we obtain

$$M_1^{(k)}(n) = \frac{(\alpha_0-1)\left(2\alpha_0^k-1-k\alpha_0\right)}{\left[2\alpha_0^k-(k+1)\right]^2}n\alpha_0^n + O(\alpha_0^n).$$

□

We also find an approximate formula for $M_2^{(k)}(n)$.

Theorem 6.3. *Let $k \geq 2$ and $n \geq k$ be integers. Let α_0 be as defined in Lemma 6.1. Then*

$$M_2^{(k)}(n) = \frac{\left(1 - \frac{1}{\alpha_0}\right)\left[2\alpha_0^k - 1 - k\alpha_0\right]^2}{\left[2\alpha_0^k - (k+1)\right]^3} n^2\alpha_0^n + O(n\alpha_0^n).$$

Proof. We begin with the definition of $M_2^{(k)}$,

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1}\left(M_2^{(k)}(n-i-1) + 2iM_1^{(k)}(n-i-1) + i^2F_{n-i-1}^{(k)}\right),$$

that we reindex to get

$$M_2^{(k)}(n) = \sum_{i=1}^{k}\left(M_2^{(k)}(n-i) + 2(i-1)M_1^{(k)}(n-i) + (i-1)^2F_{n-i}^{(k)}\right).$$

Define the generating function $G_2(z)$ as

$$G_2(z) = \sum_{n\geq 0} M_2^{(k)}(n)z^n.$$

Then substituting the previous recursive definition of $M_2^{(k)}(n)$ we get

$$G_2(z) = \sum_{n\geq 0}\left[\sum_{i=1}^{k}\left(M_2^{(k)}(n-i) + 2(i-1)M_1^{(k)}(n-i) + (i-1)^2F_{n-i}^{(k)}\right)\right]z^n.$$

Distributing the z^n, splitting up the sums, reordering them, and factoring $2(i-1)$ out of the center inner sum, and $(i-1)^2$ out of the right inner sum we obtain

$$\begin{aligned} G_2(z) &= \sum_{i=1}^{k}\sum_{n\geq 0} M_2^{(k)}(n-i)z^n + \sum_{i=1}^{k}\left[2(i-1)\sum_{n\geq 0} M_1^{(k)}(n-i)z^n\right] \\ &+ \sum_{i=1}^{k}\left[(i-1)^2\sum_{n\geq 0} F_{n-i}^{(k)}z^n\right]. \end{aligned}$$

Factoring z^i out of each inner sum and reindexing them we get

$$\begin{aligned} G_2(z) &= \sum_{i=1}^{k}\left[z^i\sum_{n\geq -i} M_2^{(k)}(n)z^n\right] + \sum_{i=1}^{k}\left[2(i-1)z^i\sum_{n\geq -i} M_1^{(k)}(n)z^n\right] \\ &+ \sum_{i=1}^{k}\left[(i-1)^2z^i\sum_{n\geq -i} F_n^{(k)}z^n\right]. \end{aligned}$$

But since $M_2^{(k)}(n)$, $M_1^{(k)}(n)$, and $F_n^{(k)}$ are zero for $n<0$, we can delete these terms to get

$$\begin{aligned} G_2(z) &= \sum_{i=1}^{k}\left[z^i\sum_{n\geq 0}M_2^{(k)}(n)z^n\right]+\sum_{i=1}^{k}\left[2(i-1)z^i\sum_{n\geq 0}M_1^{(k)}(n)z^n\right] \\ &+ \sum_{i=1}^{k}\left[(i-1)^2z^i\sum_{n\geq 0}F_n^{(k)}z^n\right]. \end{aligned}$$

Now we can substitute $G_2(z)$ for $\sum\limits_{n\geq 0}M_2^{(k)}(n)z^n$, $G_1(z)$ for $\sum\limits_{n\geq 0}M_1^{(k)}(n)z^n$, and $F(z)$ for $\sum\limits_{n\geq 0}F_n^{(k)}z^n$, to get

$$G_2(z)=\sum_{i=1}^{k}z^iG_2(z)+\sum_{i=1}^{k}2(i-1)z^iG_1(z)+\sum_{i=1}^{k}(i-1)^2z^iF(z).$$

Solving for $G_2(z)$ we obtain

$$G_2(z)=\frac{2G_1(z)\sum_{i=1}^{k}(i-1)z^i+F(z)\sum_{i=1}^{k}(i-1)^2z^i}{1-\sum_{i=1}^{k}z^i}.$$

To complete the generating function $G_2(z)$ we substitute $G_1(z)=\frac{z^{k-1}\sum_{i=1}^{k}(i-1)z^i}{\left(1-\sum_{i=1}^{k}z^i\right)^2}$ and $F(z)=\frac{z^{k-1}}{1-\sum_{i=1}^{k}z^i}$ to get

$$G_2(z)=\frac{2\frac{z^{k-1}\sum_{i=1}^{k}(i-1)z^i}{\left(1-\sum_{i=1}^{k}z^i\right)^2}\sum_{i=1}^{k}(i-1)z^i+\frac{z^{k-1}}{1-\sum_{i=1}^{k}z^i}\sum_{i=1}^{k}(i-1)^2z^i}{1-\sum_{i=1}^{k}z^i}.$$

This simplifies to

$$G_2(z)=\frac{2z^{k-1}\left(\sum_{i=1}^{k}(i-1)z^i\right)^2+z^{k-1}\left(\sum_{i=1}^{k}(i-1)^2z^i\right)\left(1-\sum_{i=1}^{k}z^i\right)}{\left(1-\sum_{i=1}^{k}z^i\right)^3}.$$

Using the factorization of $1-\sum\limits_{i=1}^{k}z^i$ given in Lemma 6.1, we rewrite the right hand side to get

$$G_2(z)=\frac{2z^{k-1}\left(\sum_{i=1}^{k}(i-1)z^i\right)^2+z^{k-1}\left(\sum_{i=1}^{k}(i-1)^2z^i\right)\left(1-\sum_{i=1}^{k}z^i\right)}{\prod_{i=0}^{k-1}(1-\alpha_iz)^3}.$$

Using partial fractions we have

$$\sum_{i=0}^{k-1}\left(\frac{A_i}{1-\alpha_iz}+\frac{B_i}{(1-\alpha_iz)^2}+\frac{C_i}{(1-\alpha_iz)^3}\right),$$

and then by geometric series this is equal to

$$\sum_{i=0}^{k-1}\left(A_i\sum_{n\geq 0}(\alpha_i z)^n + B_i\sum_{n\geq 0}(n+1)(\alpha_i z)^n + C_i\sum_{n\geq 0}\frac{(n+1)(n+2)}{2}(\alpha_i z)^n\right).$$

Since we defined $G_2(z) = \sum_{n\geq 0} M_2^{(k)}(n)z^n$ we have

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1}\left(A_i\alpha_i^n + B_i(n+1)\alpha_i^n + C_i\frac{(n+1)(n+2)}{2}\alpha_i^n\right).$$

This becomes

$$M_2^{(k)}(n) = \sum_{i=0}^{k-1}\left(\left(A_i + B_i(n+1) + C_i\frac{(n+1)(n+2)}{2}\right)\alpha_i^n\right).$$

Now the only term of significance as $n \to \infty$ on the right hand of this equation is the term with α_0 since $1 < \alpha_0 < 2$ and $\|\alpha_i\| < 1$ for all $i \neq 0$. Of that term only the constant C_0 matters as $n \to \infty$ since it is multiplied by $\frac{(n+1)(n+2)}{2}$, B_0 is multiplied by $(n+1)$, and A_0 is just a constant. Therefore, we can restate the equation using the big O notation and Miller's facts about the α_i's.

$$M_2^{(k)}(n) = C_0\frac{n^2}{2}\alpha_0^n + O(n\alpha_0^n).$$

Thus, we must solve for C_0. To do that we begin with

$$\begin{aligned}&\frac{2z^{k-1}\left(\sum_{i=1}^{k}(i-1)z^i\right)^2 + z^{k-1}\left(\sum_{i=1}^{k}(i-1)^2z^i\right)\left(1-\sum_{i=1}^{k}z^i\right)}{\prod_{i=0}^{k-1}(1-\alpha_i z)^3}\\ =& \sum_{i=0}^{k-1}\left(\frac{A_i}{1-\alpha_i z} + \frac{B_i}{(1-\alpha_i z)^2} + \frac{C_i}{(1-\alpha_i z)^3}\right).\end{aligned}$$

We then multiply each side by $\prod_{i=0}^{k-1}(1-\alpha_i z)^3$ to get

$$\begin{aligned}&2z^{k-1}\left(\sum_{i=1}^{k}(i-1)z^i\right)^2 + z^{k-1}\left(\sum_{i=1}^{k}(i-1)^2z^i\right)\left(1-\sum_{i=1}^{k}z^i\right)\\ =& \sum_{i=0}^{k-1}\left[\left(A_i(1-\alpha_i z)^2 + B_i(1-\alpha_i z) + C_i\right)\prod_{\substack{j=0\\ j\neq i}}^{k-1}(1-\alpha_j z)^3\right].\end{aligned}$$

We can then substitute $z = \frac{1}{\alpha_0}$ and note that this makes all terms in the right sum zero except for the first term ($i = 0$); this also makes the second term on the left zero since $\frac{1}{\alpha_0}$ is a root of $1-\sum_{i=1}^{k} z^i$ and this yields

$$\begin{aligned}&2\left(\frac{1}{\alpha_0}\right)^{k-1}\left(\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^{i}\right)^{2}\\ =\;&\left[A_0\left(1-\alpha_0\frac{1}{\alpha_0}\right)^{2}+B_0\left(1-\alpha_0\frac{1}{\alpha_0}\right)+C_0\right]\prod_{\substack{j=0\\ j\neq 0}}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^{3}.\end{aligned}$$

This simplifies to

$$2\left(\frac{1}{\alpha_0}\right)^{k-1}\left(\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^{i}\right)^{2}=C_0\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^{3}.$$

Therefore,

$$C_0=\frac{2\left(\frac{1}{\alpha_0}\right)^{k-1}\left(\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^{i}\right)^{2}}{\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^{3}}.$$

Multiply the numerator and denominator by $\left(\frac{1}{\alpha_0}-1\right)^3(-\alpha_0^3)$ to obtain

$$C_0=\frac{-2\left(\frac{1}{\alpha_0}\right)^{k-4}\left(\frac{1}{\alpha_0}-1\right)^{3}\left(\sum_{i=1}^{k}(i-1)\left(\frac{1}{\alpha_0}\right)^{i}\right)^{2}}{\left(\frac{1}{\alpha_0}-1\right)^{3}(-\alpha_0^3)\prod_{j=1}^{k-1}\left(1-\alpha_j\frac{1}{\alpha_0}\right)^{3}}.$$

By applying Lemma 6.3 we obtain

$$C_0=\frac{-2\left(\frac{1}{\alpha_0}\right)^{k-4}\left(\frac{1}{\alpha_0}-1\right)\left[-2\left(\frac{1}{\alpha_0}\right)^{2}+\left(\frac{1}{\alpha_0}\right)^{k+2}+k\left(\frac{1}{\alpha_0}\right)^{k+1}\right]^{2}}{\left[2-(k+1)\left(\frac{1}{\alpha_0}\right)^{k}\right]^{3}}.$$

Simplifying and multiplying the numerator and denominator by α_0^{3k} we get

$$C_0=\frac{2\left(1-\frac{1}{\alpha_0}\right)\left[2\alpha_0^k-1-k\alpha_0\right]^2}{\left[2\alpha_0^k-(k+1)\right]^3}.$$

Therefore, if we substitute this result back into the earlier equation we obtain

$$M_2^{(k)}(n)=\frac{\left(1-\frac{1}{\alpha_0}\right)\left[2\alpha_0^k-1-k\alpha_0\right]^2}{\left[2\alpha_0^k-(k+1)\right]^3}n^2\alpha_0^n+O(n\alpha_0^n).$$

□

7. The mean and variance of the k-Zeckendorf digital sums

Now we have the proper tools to compute the mean and variance of the k-Zeckendorf digital sums up to $F_n^{(k)}$.

Note that if $x \geq 0$ is an integer, $\mu = \frac{1}{x}\sum_{0\leq k<x} f(k)$ is the mean and $\sigma^2 = \frac{1}{x}\sum_{0\leq k<x}(f(k))^2 - \mu^2$ is the variance of the set $\{f(0), f(1), f(2), \ldots, f(x-1)\}$. For convenience, we will let $x = F_n^{(k)}$ for some n.

Lemma 7.1. *Let μ_n be the mean of $\left\{z_k(0), z_k(1), z_k(2), \ldots, z_k\left(F_n^{(k)} - 1\right)\right\}$. Then*

$$\mu_n = \frac{M_1^{(k)}(n)}{F_n^{(k)}}.$$

Proof. From the above note we have that

$$\mu_n = \frac{1}{F_n^{(k)}} \sum_{0\leq i<F_n^{(k)}} z_k(i).$$

But since φ_k is defined as the quantity of numbers less than $F_n^{(k)}$ with k-Zeckendorf digital sum of m, we can rewrite this as

$$\mu_n = \frac{1}{F_n^{(k)}} \sum_{m=1}^{n} m\varphi_k(n,m).$$

But this is

$$\mu_n = \frac{M_1^{(k)}(n)}{F_n^{(k)}}.$$

□

Lemma 7.2. *Let σ_n^2 be the variance of $\left\{z_k(0), z_k(1), z_k(2), \cdots, z_k\left(F_n^{(k)} - 1\right)\right\}$. Then*

$$\sigma_n^2 = \frac{M_2^{(k)}(n)}{F_n^{(k)}} - \mu_n^2.$$

Proof. From the above note we have that

$$\sigma_n^2 = \frac{1}{F_n^{(k)}} \sum_{0\leq i<F_n^{(k)}} (z_k(i))^2 - \mu_n^2.$$

But since φ_k is defined as the quantity of numbers less than $F_n^{(k)}$ with k-Zeckendorf digital sum of m, we can rewrite this as

$$\sigma_n^2 = \frac{1}{F_n^{(k)}} \sum_{m=1}^{n} m^2\varphi_k(n,m) - \mu_n^2.$$

But this is

$$\sigma_n^2 = \frac{M_2^{(k)}(n)}{F_n^{(k)}} - \mu_n^2.$$

□

8. The Natural Density of the k-Zeckendorf Niven Numbers

Now we will give proofs for all the conditions specified in Theorem 3.1 so that we can conclude that the natural density of the k-Zeckendorf Niven numbers is zero.

Lemma 8.1. *The sequence $F_n^{(k)}$ is increasing.*

Proof. This is obvious since $F_n^{(k)}$ is the sum of $F_{n-1}^{(k)}$ and other nonnegative terms. □

Lemma 8.2. *Let μ_n be the mean of $\left\{z_k(0), z_k(1), z_k(2), \ldots, z_k\left(F_n^{(k)}-1\right)\right\}$. Then*

$$\lim_{n\to\infty} \mu_n = \infty.$$

Proof. By Lemma 7.1, we have

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{M_1^{(k)}(n)}{F_n^{(k)}}.$$

Substituting the closed forms of $M_1^{(k)}(n)$ and $F_n^{(k)}$ into the right hand side we get

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{\frac{(\alpha_0-1)\left(2\alpha_0^k-1-k\alpha_0\right)}{(2\alpha_0^k-(k+1))^2} n\alpha_0^n + O(\alpha_0^n)}{\sum_{i=0}^{k-1} \frac{\alpha_i^2-\alpha_i}{2\alpha_i^k-(k+1)}\alpha_i^n}.$$

Replacing the denominator by $\frac{\alpha_0^2-\alpha_0}{2\alpha_0^k-(k+1)}\alpha_0^n + O(1)$ and taking the limit we get

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{\frac{(\alpha_0-1)\left(2\alpha_0^k-1-k\alpha_0\right)}{\left[2\alpha_0^k-(k+1)\right]^2} n\alpha_0^n + O(\alpha_0^n)}{\frac{\alpha_0^2-\alpha_0}{2\alpha_0^k-(k+1)}\alpha_0^n + O(1)} = \infty.$$

□

Lemma 8.3. *Let μ_n and σ_n be the mean and variance of $\{z_k(0), z_k(1), z_k(2), \ldots, z_k\left(F_n^{(k)}-1\right)\}$. Then*

$$\lim_{n\to\infty} \frac{\mu_n}{\sigma_n} = \infty.$$

Proof. To prove this result we will show that $\lim_{n\to\infty} \frac{\mu_n^2}{\sigma_n^2} = \infty$. To this end we begin by using Lemma 7.1 and Lemma 7.2 to obtain

$$\frac{\mu_n^2}{\sigma_n^2} = \frac{\left(\frac{M_1^{(k)}(n)}{F_n^{(k)}}\right)^2}{\frac{M_2^{(k)}(n)}{F_n^{(k)}} - \left(\frac{M_1^{(k)}(n)}{F_n^{(k)}}\right)^2}.$$

Multiplying the numerator and denominator by $\left(F_n^{(k)}\right)^2$ we get

$$\frac{\mu_n^2}{\sigma_n^2} = \frac{\left(M_1^{(k)}(n)\right)^2}{F_n^{(k)} M_2^{(k)}(n) - \left(M_1^{(k)}(n)\right)^2}.$$

Now, multiplying the numerator and denominator by $\left[2\alpha_0^k-(k+1)\right]^4$ we have that

$$\frac{\mu_n^2}{\sigma_n^2}=\frac{\left[2\alpha_0^k-(k+1)\right]^4\left(M_1^k(n)\right)^2}{\left[2\alpha_0^k-(k+1)\right]^4F_n^{(k)}M_2^{(k)}(n)-\left[2\alpha_0^k-(k+1)\right]^4\left(M_1^{(k)}(n)\right)^2}.$$

Now we substitute the closed forms of $F_n^{(k)}$, $M_1^{(k)}(n)$, and $M_2^{(k)}(n)$ into this last expression and simplify. These closed forms are

$$\begin{aligned}
F_n^{(k)} &= \frac{\alpha_0^2-\alpha_0}{2\alpha_0^k-(k+1)}\alpha_0^n+O(1),\\
M_1^{(k)}(n) &= \frac{(\alpha_0-1)(2\alpha_0^k-1-k\alpha_0)}{(2\alpha_0^k-(k+1))^2}n\alpha_0^n+O(\alpha_0^n),\\
\text{and } M_2^{(k)}(n) &= \frac{(1-\frac{1}{\alpha_0})(2\alpha_0^k-1-k\alpha_0)^2}{(2\alpha_0^k-(k+1))^3}n^2\alpha_0^n+O(n\alpha_0^n).
\end{aligned}$$

It follows that

$$\frac{\mu_n^2}{\sigma_n^2}$$

$$=\frac{(\alpha_0-1)^2(2\alpha_0^k-1-k\alpha_0)^2n^2\alpha_0^{2n}+O(n\alpha_0^{2n})}{(\alpha_0^2-\alpha_0)(1-\frac{1}{\alpha_0})(2\alpha_0^k-1-k\alpha_0)^2n^2\alpha_0^n+O(n\alpha_0^{2n})-(\alpha_0-1)^2(2\alpha_0^k-1-k\alpha_0)^2n^2\alpha_0^{2n}+O(n\alpha_0^{2n})}$$

$$=\frac{(\alpha_0-1)^2(2\alpha_0^k-1-k\alpha_0)^2n^2\alpha_0^{2n}+O(n\alpha_0^{2n})}{O(n\alpha_0^{2n})}.$$

Taking the limit of the last expression as $n\to\infty$ we obtain

$$\lim_{n\to\infty}\frac{\mu_n^2}{\sigma_n^2}=\lim_{n\to\infty}\frac{(\alpha_0-1)^2(2\alpha_0^k-1-k\alpha_0)^2n^2\alpha_0^{2n}+O(n\alpha_0^{2n})}{O(n\alpha_0^{2n})}=\infty.$$

□

Lemma 8.4. *The sequence* $\frac{F_{n+1}^{(k)}}{F_n^{(k)}}$ *is bounded.*

Proof.

$$\frac{F_{n+1}^{(k)}}{F_n^{(k)}}=\frac{\left[F_n^{(k)}+F_{n-1}^{(k)}+\cdots+F_{n-k+1}^{(k)}\right]+F_{n-k}^{(k)}-F_{n-k}^{(k)}}{F_n^{(k)}}=\frac{2F_n^{(k)}-F_{n-k}^{(k)}}{F_n^{(k)}}\leq\frac{2F_n^{(k)}}{F_n^{(k)}}=2$$

□

We can now state the following major result.

Theorem 8.1. *The natural density of the k-Zeckendorf Niven numbers is* 0.

Proof. To prove this, we apply Theorem 3.1, noting that all of the conditions for its use have been shown in the previous lemmas. □

9. Open Questions

We conclude this paper with some open questions. In the proof of Theorem 6.2 we computed the constant B_0. Because of our error analysis we did not need to calculate $A_0, A_1, \ldots, A_{k-1}$ and $B_1, B_2, \ldots, B_{k-1}$. Although the calculation of $B_1, B_2, \ldots, B_{k-1}$ would be similar, an open question is to determine exact formulas for $A_0, A_1, \ldots, A_{k-1}$.

Similarly, in the proof of Theorem 6.3 we computed the constant C_0. Although the calculation of $C_1, C_2, \ldots, C_{k-1}$ would be similar, an open question is to determine exact formulas for $A_0, A_1, \ldots, A_{k-1}$ and $B_0, B_1, \ldots, B_{k-1}$. Note that these are different from the constants in Theorem 6.2.

Finally, we proved here that the natural density of the k-Zeckendorf Niven numbers is zero. To study how many k-Zeckendorf Niven numbers there are, we ask to find an asymptotic formula for $Z_k(n)$, the number of k-Zeckendorf Niven numbers less than n. That is, can we find a non-trivial function $f(n)$ such that

$$Z_k(n) \sim f(n).$$

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