FINDING MOMENTS OF THE LARGE DIGIT FUNCTION USING DELANGE'S METHOD

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Abstract. Let $L_b(i)$ denote the number of large digits ($\lceil b/2 \rceil$ or more) in the base b representation of the nonnegative integer i. We will use Delange's method to find moments of the large digit function.

1. Introduction. Let $s_b(i)$ denote the sum of the digits in the base b representation of the nonnegative integer i and $L_b(i)$ denote the number of large digits ($\lceil b/2 \rceil$ or more) in the base b representation of the nonnegative integer i. Delange [5] and later Grabner, Kirschenhofer, Prodinger, and Tichy [6] proved that

$$\sum_{n<N} s_b(n) = \frac{b-1}{2} N \log_b N + N\delta(\log_b N),$$

where $\delta(u)$ is a fractal function and $\log_b N$ denotes the base b logarithm of N. Coquet [4] and Kirschenhofer [7] proved that

$$\sum_{n<N} s_b(n)^2 = \left(\frac{b-1}{2}\right)^2 N \log_b^2 N + N \log_b N \delta_1(\log_b N) + N\delta_2(\log_b N),$$

where δ_1 and δ_2 are continuous nowhere differentiable functions of period 1. Grabner, Kirschenhofer, Prodinger, and Tichy also found higher moments of the sum-of-digits function. Cooper and Kennedy [3] computed the first moment of the large digit function using a method of Trollope [10]. Here, we will use Delange's method to find moments of the large digit function. Our result will be quite similar to the results of Coquet, Kirschenhofer, and Grabner, Kirschenhofer, Prodinger, and Tichy. We will conclude the paper with some open questions.

2. First Moment. Let

$$M_1(N) = \sum_{n<N} L_b(n).$$

For $j \geq 0$, let

$$f_j(t) = \left\lfloor \frac{t}{b^{j+1}} + \frac{\lfloor b/2 \rfloor}{b} \right\rfloor - \left\lfloor \frac{t}{b^{j+1}} \right\rfloor$$

and

$$g_j(t) = f_j(t) - \frac{\lfloor b/2 \rfloor}{b}.$$

Let $L = \lfloor \log_b N \rfloor$. Then,

$$\begin{aligned} M_1(N) &= \int_0^N \sum_{j=0}^{L} f_j(t)dt \\ &= \int_0^N \sum_{j=0}^{L} g_j(t)dt + \frac{\lfloor b/2 \rfloor}{b} N(L+1) \\ &= \frac{\lfloor b/2 \rfloor}{b} N(L+1) + \sum_{j=0}^{L} \int_0^N g_j(t)dt. \end{aligned}$$

Now letting

$$u = \frac{t}{b^{j+1}},$$

and

$$g_j(t) = g(u) = \left\lfloor u + \frac{\lfloor b/2 \rfloor}{b} \right\rfloor - \lfloor u \rfloor - \frac{\lfloor b/2 \rfloor}{b},$$

we have

$$M_1(N) = \frac{\lfloor b/2 \rfloor}{b} N(L+1) + \sum_{j=0}^{L} b^{j+1} \int_0^{N/b^{j+1}} g(u)du.$$

Letting $k = L - j$ it follows that

$$M_1(N) = \frac{\lfloor b/2 \rfloor}{b} N(L+1) + \sum_{k=0}^{L} b^{L+1-k} \int_0^{N/b^{L+1-k}} g(u)du.$$

But for each $k > L$,

$$\int_0^{N/b^{L+1-k}} g(u)du = 0.$$

Since $L = \lfloor \log_b N \rfloor$ and $\{\log_b N\} = \log_b N - \lfloor \log_b N \rfloor$,

$$\frac{N}{b^{\lfloor \log_b N \rfloor + 1 - k}} = \frac{N}{b^{\log_b N - \{\log_b N\} + 1 - k}} = b^{k + \{\log_b N\} - 1}.$$

Therefore,

$$M_1(N) = \frac{\lfloor b/2 \rfloor}{b} N(L+1) + b^{L+1} \sum_{k \geq 0} b^{-k} \int_0^{b^{k+\{\log_b N\}-1}} g(u)du.$$

Since

$$L + 1 = \lfloor \log_b N \rfloor + 1 = \log_b N - \{\log_b N\} + 1,$$

it follows that

$$\begin{aligned} M_1(N) = &\frac{\lfloor b/2 \rfloor}{b} N\Big(\log_b N - \{\log_b N\} + 1\Big) \\ &+ N \cdot b^{1-\{\log_b N\}} \sum_{k \geq 0} b^{-k} \int_0^{b^{k+\{\log_b N\}-1}} g(u)du. \end{aligned}$$

Next, let

$$h_1(x) = \int_0^x g(u)du.$$

Note that $h_1(0) = h_1(1) = 0$. Then

$$M_1(N) = \frac{\lfloor b/2 \rfloor}{b} N\Big(\log_b N - \{\log_b N\} + 1\Big) + N \cdot b^{1-\{\log_b N\}} \sum_{k \geq 0} b^{-k} h_1(b^{k+\{\log_b N\}-1}).$$

Now letting

$$\psi_1(x) = b^{1-\{x\}} \sum_{k \geq 0} b^{-k} h_1(b^k b^{\{x\}-1}),$$

it follows that

$$M_1(N) = \frac{\lfloor b/2 \rfloor}{b} N\Big(\log_b N - \{\log_b N\} + 1\Big) + N\psi_1(\log_b N).$$

Finally, letting

$$\delta_1(x) = \frac{\lfloor b/2 \rfloor}{b}\Big(1 - \{x\}\Big) + \psi_1(x),$$

we have that

$$\begin{aligned} M_1(N) &= \frac{\lfloor b/2 \rfloor}{b} N \log_b N + N \cdot \left(\frac{\lfloor b/2 \rfloor}{b}\Big(1 - \{\log_b N\}\Big) + \psi_1(\log_b N)\right) \\ &= \frac{\lfloor b/2 \rfloor}{b} N \log_b N + N\delta_1(\log_b N). \end{aligned}$$

3. Second Moment. Let

$$M_2(N) = \sum_{n<N} L_b(n)^2.$$

For $j \geq 0$, let

$$f_j(t) = \left\lfloor \frac{t}{b^{j+1}} + \frac{\lfloor b/2 \rfloor}{b} \right\rfloor - \left\lfloor \frac{t}{b^{j+1}} \right\rfloor$$

and

$$g_j(t) = f_j(t) - \frac{\lfloor b/2 \rfloor}{b}.$$

Let $L = \lfloor \log_b N \rfloor$. Then,

$$\begin{aligned} M_2(N) &= \int_0^N \sum_{0 \leq j_1, j_2 \leq L} f_{j_1}(t) f_{j_2}(t) dt \\ &= \int_0^N \sum_{0 \leq j_1, j_2 \leq L} \left(\frac{\lfloor b/2 \rfloor}{b} + g_{j_1}(t) \right) \left(\frac{\lfloor b/2 \rfloor}{b} + g_{j_2}(t) \right) dt \\ &= \int_0^N \sum_{0 \leq j_1, j_2 \leq L} \left(\frac{\lfloor b/2 \rfloor^2}{b^2} + \frac{\lfloor b/2 \rfloor}{b} g_{j_1}(t) + \frac{\lfloor b/2 \rfloor}{b} g_{j_2}(t) + g_{j_1}(t) g_{j_2}(t) \right) dt. \end{aligned}$$

Letting

$$I_2 = \int_0^N \sum_{0 \leq j_1, j_2 \leq L} g_{j_1}(t) g_{j_2}(t) dt,$$

we have that

$$M_2(N) = I_2 + \frac{2\lfloor b/2 \rfloor}{b}(L+1) \int_0^N \sum_{j=0}^{L} g_j(t) dt + \frac{\lfloor b/2 \rfloor^2}{b^2} N(L+1)^2$$

$$= I_2 + \frac{2\lfloor b/2 \rfloor}{b}(L+1) \int_0^N \sum_{j=0}^{L} f_j(t) dt - \frac{2\lfloor b/2 \rfloor^2}{b^2} N(L+1)^2 + \frac{\lfloor b/2 \rfloor^2}{b^2} N(L+1)^2$$

$$= I_2 + \frac{2\lfloor b/2 \rfloor}{b}(L+1) M_1(N) - \frac{\lfloor b/2 \rfloor^2}{b^2} N(L+1)^2.$$

Letting

$$K = \log_b N, \qquad y = 1 - \{\log_b N\}, \qquad \delta_1 = \delta_1(\log_b N),$$

where δ_1 comes from the first moment, it follows that

$$\begin{aligned}
M_2(N) &= I_2 + \frac{2\lfloor b/2\rfloor}{b}(L+1)M_1(N) - \frac{\lfloor b/2\rfloor^2}{b^2}N(L+1)^2 \\
&= I_2 + \frac{2\lfloor b/2\rfloor}{b}(K+y)\left(\frac{\lfloor b/2\rfloor}{b}NK + N\delta_1\right) - \frac{\lfloor b/2\rfloor^2}{b^2}N(K+y)^2 \\
&= I_2 + \frac{2\lfloor b/2\rfloor^2}{b^2}NK^2 + \frac{2\lfloor b/2\rfloor}{b}NK\delta_1 + \frac{2\lfloor b/2\rfloor^2}{b^2}NKy + \frac{2\lfloor b/2\rfloor}{b}Ny\delta_1 \\
&\quad - \frac{\lfloor b/2\rfloor^2}{b^2}NK^2 - \frac{2\lfloor b/2\rfloor^2}{b^2}NKy - \frac{\lfloor b/2\rfloor^2}{b^2}Ny^2 \\
&= I_2 + \frac{\lfloor b/2\rfloor^2}{b^2}NK^2 + \frac{2\lfloor b/2\rfloor}{b}NK\delta_1 + \frac{2\lfloor b/2\rfloor}{b}Ny\delta_1 - \frac{\lfloor b/2\rfloor^2}{b^2}Ny^2.
\end{aligned}$$

Now we compute I_2 as

$$\begin{aligned}
I_2 &= \int_0^N \sum_{0\le j_1,j_2\le L} g_{j_1}(t)g_{j_2}(t)dt \\
&= \left(\sum_{0\le j_1=j_2\le L} + 2\sum_{0\le j_1<j_2\le L}\right)\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \sum_{j=0}^{L}\int_0^N g_j(t)^2dt + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \sum_{j=0}^{L}\int_0^N \left(f_j(t) - \frac{\lfloor b/2\rfloor}{b}\right)^2 dt + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \sum_{j=0}^{L}\int_0^N \left(\left(1 - \frac{2\lfloor b/2\rfloor}{b}\right)f_j(t) + \frac{\lfloor b/2\rfloor^2}{b^2}\right)dt + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \left(1 - \frac{2\lfloor b/2\rfloor}{b}\right)M_1(N) + \frac{\lfloor b/2\rfloor^2}{b^2}N(L+1) + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \left(1 - \frac{2\lfloor b/2\rfloor}{b}\right)M_1(N) + \frac{\lfloor b/2\rfloor^2}{b^2}N(K+y) + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt \\
&= \left(1 - \frac{2\lfloor b/2\rfloor}{b}\right)M_1(N) + \frac{\lfloor b/2\rfloor^2}{b^2}NK + \frac{\lfloor b/2\rfloor^2}{b^2}Ny \\
&\quad + 2\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt.
\end{aligned}$$

Now let

$$u = \frac{t}{b^{j_2+1}}.$$

Then $dt = b^{j_2+1}du$ and

$$g_{j_2}(t) = g(u) = \left\lfloor u + \frac{\lfloor b/2 \rfloor}{b} \right\rfloor - \lfloor u \rfloor - \frac{\lfloor b/2 \rfloor}{b}.$$

Thus,

$$\sum_{0 \le j_1 < j_2 \le L} \int_0^N g_{j_1}(t) g_{j_2}(t) dt$$
$$= \sum_{0 \le j_1 < j_2 \le L} b^{j_2+1} \int_0^{N/b^{j_2+1}} g(b^{j_2-j_1}u) g(u) du.$$

Letting $j = j_2$, $d = j_2 - j_1$, and $k = L - j$, it follows that

$$\sum_{0 \le j_1 < j_2 \le L} \int_0^N g_{j_1}(t) g_{j_2}(t) dt$$
$$= \sum_{j=0}^{L} b^{j+1} \sum_{1 \le d \le j} \int_0^{N/b^{j+1}} g(b^d u) g(u) du$$
$$= \sum_{k=0}^{L} b^{L+1-k} \sum_{1 \le d \le L-k} \int_0^{N/b^{L+1-k}} g(b^d u) g(u) dt.$$

But for each $k > L$ and $d \ge 1$ or $0 \le k \le L$ and $d > L - k$,

$$\int_0^{N/b^{L+1-k}} g(b^d u) g(u) du = 0.$$

Since $L = \lfloor \log_b N \rfloor$,

$$\frac{N}{b^{\lfloor \log_b N \rfloor + 1 - k}} = \frac{N}{b^{\log_b N - \{\log_b N\} + 1 - k}} = b^{k + \{\log_b N\} - 1}.$$

Therefore,

$$\sum_{0 \le j_1 < j_2 \le L} \int_0^N g_{j_1}(t) g_{j_2}(t) dt$$
$$= b^{L+1} \sum_{k \ge 0} b^{-k} \sum_{1 \le d} \int_0^{b^{k+\{\log_b N\}-1}} g(b^d u) g(u) du.$$

Since

$$L + 1 = \lfloor \log_b N \rfloor + 1 = \log_b N - \{\log_b N\} + 1,$$

it follows that

$$\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt$$
$$= N\cdot b^{1-\{\log_b N\}}\sum_{k\ge 0} b^{-k}\sum_{1\le d}\int_0^{b^{k+\{\log_b N\}-1}} g(b^d u)g(u)du.$$

Next, let

$$h_2(x) = \sum_{1\le d}\int_0^x g(b^d u)g(u)du.$$

Note that $h_2(0) = h_2(1) = 0$. Then

$$\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt$$
$$= N\cdot b^{1-\{\log_b N\}}\sum_{k\ge 0} b^{-k}h_2(b^{k+\{\log_b N\}-1}).$$

Now letting

$$\psi_2(x) = b^{1-\{x\}}\sum_{k\ge 0} b^{-k}h_2(b^k b^{\{x\}-1}),$$

it follows that

$$\sum_{0\le j_1<j_2\le L}\int_0^N g_{j_1}(t)g_{j_2}(t)dt = N\psi_2(\log_b N).$$

Thus, letting $\psi_2 = \psi_2(\log_b N)$ and substituting for I_2, $M_1(N)$, the integral, and

$K = \log_b N$, we have

$$
\begin{aligned}
M_2(N) &= I_2 + \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 + \frac{2\lfloor b/2 \rfloor}{b} NK\delta_1 + \frac{2\lfloor b/2 \rfloor}{b} Ny\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} Ny^2 \\
&= \left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right) M_1(N) + \frac{\lfloor b/2 \rfloor^2}{b^2} NK + \frac{\lfloor b/2 \rfloor^2}{b^2} Ny \\
&\quad + 2 \sum_{0 \le j_1 < j_2 \le L} \int_0^N g_{j_1}(t) g_{j_2}(t) dt + \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 \\
&\quad + \frac{2\lfloor b/2 \rfloor}{b} NK\delta_1 + \frac{2\lfloor b/2 \rfloor}{b} Ny\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} Ny^2 \\
&= \left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right)\left(\frac{\lfloor b/2 \rfloor}{b} NK + N\delta_1\right) \\
&\quad + \frac{\lfloor b/2 \rfloor^2}{b^2} NK + \frac{\lfloor b/2 \rfloor^2}{b^2} Ny + 2 \sum_{0 \le j_1 < j_2 \le L} \int_0^N g_{j_1}(t) g_{j_2}(t) dt \\
&\quad + \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 + \frac{2\lfloor b/2 \rfloor}{b} NK\delta_1 + \frac{2\lfloor b/2 \rfloor}{b} Ny\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} Ny^2 \\
&= \left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right)\left(\frac{\lfloor b/2 \rfloor}{b} NK + N\delta_1\right) + \frac{\lfloor b/2 \rfloor^2}{b^2} NK + \frac{\lfloor b/2 \rfloor^2}{b^2} Ny + 2N\psi_2 \\
&\quad + \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 + \frac{2\lfloor b/2 \rfloor}{b} NK\delta_1 + \frac{2\lfloor b/2 \rfloor}{b} Ny\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} Ny^2 \\
&= \left(\frac{\lfloor b/2 \rfloor}{b} - \frac{2\lfloor b/2 \rfloor^2}{b^2}\right) NK + \left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right) N\delta_1 + \frac{\lfloor b/2 \rfloor^2}{b^2} NK + \frac{\lfloor b/2 \rfloor^2}{b^2} Ny \\
&\quad + 2N\psi_2 + \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 + \frac{2\lfloor b/2 \rfloor}{b} NK\delta_1 + \frac{2\lfloor b/2 \rfloor}{b} Ny\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} Ny^2.
\end{aligned}
$$

Collecting terms, we have that

$$
\begin{aligned}
M_2(N) = \frac{\lfloor b/2 \rfloor^2}{b^2} NK^2 + NK\left(\left(\frac{\lfloor b/2 \rfloor}{b} - \frac{\lfloor b/2 \rfloor^2}{b^2}\right) + \frac{2\lfloor b/2 \rfloor}{b}\delta_1\right) \\
+ N\left(\left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right)\delta_1 + \frac{\lfloor b/2 \rfloor^2}{b^2} y + \frac{2\lfloor b/2 \rfloor}{b} y\delta_1 - \frac{\lfloor b/2 \rfloor^2}{b^2} y^2 + 2\psi_2\right).
\end{aligned}
$$

Finally, letting

$$
\delta_{21}(x) = \left(\frac{\lfloor b/2 \rfloor}{b} - \frac{\lfloor b/2 \rfloor^2}{b^2}\right) + \frac{2\lfloor b/2 \rfloor}{b}\delta_1(x)
$$

and

$$
\begin{aligned}
\delta_{22}(x) = \left(1 - \frac{2\lfloor b/2 \rfloor}{b}\right)\delta_1(x) + \frac{\lfloor b/2 \rfloor^2}{b^2}\Big(1 - \{x\}\Big) \\
+ \frac{2\lfloor b/2 \rfloor}{b}\Big(1 - \{x\}\Big)\delta_1(x) - \frac{\lfloor b/2 \rfloor^2}{b^2}\Big(1 - \{x\}\Big)^2 + 2\psi_2(x),
\end{aligned}
$$

it follows that

$$M_2(N) = \frac{\lfloor b/2 \rfloor^2}{b^2} N \log_b^2 N + N \log_b N \delta_{21}(\log_b N) + N\delta_{22}(\log_b N).$$

4. Questions. Some open questions remain. What can be said about higher moments of the L_b function using Delange's method? For example, what are the third and fourth moments of the L_b function? In addition, is the main term of the jth moment a constant times N times $\log_b^j N$? If so, what is this constant as a function of b? We now state the following conjecture.

Conjecture. Let m be a positive integer. Then

$$M_m(N) = \frac{\lfloor b/2 \rfloor^m}{b^m} N \log_b^m N + \sum_{k=0}^{m-1} N \log_b^k N \delta_k(\log_b N),$$

where δ_k is a fractal function for $k = 0, \ldots, m-1$.
Finally, Grabner's method [6] finds moments of the sum-of-digits function using the Mellin transform. Can Grabner's method be used to find moments of the large digit function?

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