A GENERALIZATION OF A RESULT BY NARKIEWICZ CONCERNING LARGE DIGITS OF POWERS

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1. Introduction. For some time we have been concerned about the validity of the statement

$$\text{“}s(2^n) < 2n \text{ for all } n > 3.\text{”}$$

Here, $s(m)$ denotes the (base 10) digital sum of the integer m. We have been unable to prove or disprove the above conjecture. But, based on the relation

$$s(2^n) = 2s(2^{n-1}) - 9L(2^{n-1})$$

where

$$L(m) = \#\text{ of “large digits” of } m$$

we have become interested in the number of large digits occurring in powers of 2.

Generalizing the concepts above, a “base b large digit” is a base b digit when doubled involves a “carry.” For example, 6 is a base ten large digit since $2 \times 6 = 12$ has <u>two</u> digits. The base b large digits are the digits

$$\lceil b/2 \rceil,\ \lceil b/2 \rceil + 1,\ \lceil b/2 \rceil + 2,\ \ldots\ ,\ b - 1.$$

So, for example, the base 7 large digits are

$$4,\ 5,\ \text{and } 6.$$

Digits which are not large digits are called, “small base b digits.” Hence the small base 7 digits are

$$0,\ 1,\ 2,\ \text{and } 3.$$

In general, $L_b(m)$ denotes the number of base b large digits of the base b representation of the integer m.

A method due to Narkiewicz [2] and earlier by Gupta [1] was used in investigating the question asked by Erdős:

"Does there exists an integer $m \neq 0, 2, 8$ such that

2^m is a distinct sum of powers of 3?"

That is,

"Is it true that $L_3(2^m) \neq 0$ for $m \geq 9$?"

In Narkiewicz [2], it was shown that the number of nonnegative integers $m \leq x$ such that 2^m is the sum of distinct powers of 3 does not exceed

$$1.62x^{.631}.$$

This method will be generalized in what follows.

2. Main Result.

<u>Theorem</u>. Let $(a, b) = 1$, that is a and b are relatively prime. Let

$$S = \{n : L_b(a^n) = 0\} \text{ and } S(x) = \#\{n \leq x : L_b(a^n) = 0\}.$$

Let

$$\beta = \#\{d < b : (d, b) = 1 \text{ and } d \text{ is small}\}$$

and

$$\theta = \frac{\ln\left(\frac{b+1}{2}\right)}{\ln b}.$$

Finally, let ϕ denote Euler's phi function. Then

$$S(x) \leq \beta \left(\frac{b(1-\theta)}{\theta\phi(b)}\right)^{\theta} \left(\frac{1}{1-\theta}\right) x^{\theta}.$$

Proof. Let

$$a^n = \sum_{i=0}^{s} d_i b^i.$$

If $L_b(a^n) = 0$, then $0 \le d_i \le \lfloor (b-1)/2 \rfloor$ for all $0 \le i \le s$. Then for any $k \ge 0$ (we will assume $k \ge 1$)

$$a^n \equiv \sum_{i=0}^{k-1} d_i b^i \pmod{b^k}.$$

Note that $d_0 \ne 0$, since $(a,b) = 1$. In fact, $(d_0, b) = 1$, since $(a,b) = 1$. By the definition of β, $\beta \le \phi(b)$. So,

$$\#\{a^n \pmod{b^k}\} \le \beta \left(\left\lfloor \frac{b-1}{2} \right\rfloor + 1 \right)^{k-1},$$

since $(a,b) = 1$.

The number of residue classes which n can belong to mod $\phi(b^k)$ is at most

$$\beta \left(\left\lfloor \frac{b-1}{2} \right\rfloor + 1 \right)^{k-1},$$

say,

$$r_1,\ r_2,\ \ldots,\ r_{\beta(\lfloor (b-1)/2 \rfloor)^{k-1}}.$$

Hence, for any x,

$$\#\{n \le x : n \equiv r_i \pmod{\phi(b^k)}\} \le \frac{x}{\phi(b^k)} + 1$$

and so,

$$\begin{aligned} S(x) &\le \sum_{i=1}^{\beta(\lfloor (b-1)/2 \rfloor + 1)^{k-1}} \#\{n \le x : n \equiv r_i \pmod{\phi(b^k)}\} \\ &\le \beta \left(\left\lfloor \frac{b-1}{2} \right\rfloor + 1 \right)^{k-1} \left(\frac{x}{\phi(b^k)} + 1 \right) \\ &\le \beta \left(\frac{b+1}{2} \right)^{k-1} \left(\frac{x}{\phi(b^k)} + 1 \right) \\ &= \frac{\beta}{2^{k-1}} (b+1)^{k-1} \frac{1}{b^k} \left(\frac{b^k x}{\phi(b^k)} \right) + \beta \left(\frac{b+1}{2} \right)^{k-1} \\ &= \frac{\beta}{2^{k-1}} \left(\frac{b+1}{b} \right)^{k-1} \frac{1}{b} \cdot \frac{bx}{\phi(b)} + \beta \left(\frac{b+1}{2} \right)^{k-1} \\ &= \frac{\beta}{\phi(b)} \left(\frac{b+1}{2b} \right)^{k-1} x + \beta \left(\frac{b+1}{2} \right)^{k-1}. \end{aligned}$$

Let

$$f(z) = \frac{\beta}{\phi(b)}\left(\frac{b+1}{2b}\right)^{z-1} x + \beta\left(\frac{b+1}{2}\right)^{z-1}$$

so,

$$f'(z) = \frac{\beta}{\phi(b)}\left(\frac{b+1}{2b}\right)^{z-1} x \ln\left(\frac{b+1}{2b}\right) + \beta\left(\frac{b+1}{2}\right)^{z-1} \ln\left(\frac{b+1}{2}\right)$$

and thus, $f'(z) = 0$ implies that

$$\beta\left(\frac{b+1}{2}\right)^{z-1}\left[\frac{1}{\phi(b)} \cdot \frac{x}{b^{z-1}} \ln\left(\frac{b+1}{2b}\right) + \ln\left(\frac{b+1}{2}\right)\right] = 0$$

when

$$\begin{aligned}\frac{x}{b^{z-1}} &= \frac{-\ln\left(\frac{b+1}{2}\right)}{\ln\left(\frac{b+1}{2b}\right)} \cdot \phi(b) \\ &= \frac{\ln\left(\frac{b+1}{2}\right)}{\ln\left(\frac{2b}{b+1}\right)}\phi(b).\end{aligned}$$

Since

$$\theta = \frac{\ln\left(\frac{b+1}{2}\right)}{\ln b},$$

$$\ln\left(\frac{b+1}{2}\right) = \theta \ln b$$

and

$$\begin{aligned}\ln\left(\frac{2b}{b+1}\right) &= -\ln\left(\frac{b+1}{2b}\right) \\ &= -\ln\left(\frac{b+1}{2}\right) + \ln b \\ &= -\theta \ln b + \ln b,\end{aligned}$$

and so,

$$\begin{aligned}\frac{x}{b^{z-1}} &= \frac{\theta \ln b}{-\theta \ln b + \ln b} \cdot \phi(b) \\ &= \frac{\theta}{1-\theta}\phi(b).\end{aligned}$$

Since

$$b^z \geq b^{\lfloor z \rfloor} = b^{\lceil z \rceil - 1} \geq b^{z-1},$$

$$\begin{aligned}\frac{1}{b} \cdot \frac{x}{\phi(b) b^{z-1}} &\leq \frac{x}{\phi(b) b^{\lfloor z \rfloor}} = \frac{x}{\phi(b) b^{\lceil z \rceil - 1}} \\ &\leq \frac{x}{\phi(b) b^{z-1}} = \frac{\theta}{1-\theta}.\end{aligned}$$

Therefore,

$$\frac{1}{b} \cdot \frac{\theta}{1-\theta} \le \frac{x}{\phi(b) b^{\lceil z \rceil - 1}} \le \frac{\theta}{1-\theta}.$$

Hence, there exists a k $(= \lceil z \rceil)$ that can be used. So let $k = \lceil z \rceil$ and we have

$$\frac{1}{b} \cdot \frac{\theta}{1-\theta} \le \frac{x}{\phi(b) b^{k-1}} \le \frac{\theta}{1-\theta}.$$

This implies

$$\frac{1}{b} \cdot \frac{\theta}{1-\theta} \le \frac{bx}{\phi(b) b^{k}} \le \frac{\theta}{1-\theta}.$$

This implies

$$\frac{1}{b} \cdot \frac{\theta}{1-\theta} \le \frac{x}{\phi(b^{k})} \le \frac{\theta}{1-\theta}.$$

Therefore,

$$\begin{aligned} S(x) &\le \beta\left(\frac{b+1}{2}\right)^{k-1}\left(\frac{\theta}{1-\theta}+1\right) \\ &= \beta\left(\frac{b+1}{2}\right)^{k-1}\left(\frac{1}{1-\theta}\right). \end{aligned}$$

But

$$\left(\frac{b+1}{2}\right)^{k-1} = (b^{k-1})^{\theta},$$

and

$$b^{k-1} \le \frac{bx(1-\theta)}{\theta\phi(b)},$$

so

$$\left(\frac{b+1}{2}\right)^{k-1} \le \left(\frac{b(1-\theta)}{\theta\phi(b)}\right)^{\theta} x^{\theta},$$

and we have

$$S(x) \le \beta\left(\frac{b(1-\theta)}{\theta\phi(b)}\right)^{\theta}\left(\frac{1}{1-\theta}\right) x^{\theta}.$$

This completes the proof.

Note that

$$\theta = \frac{\ln\left(\frac{b+1}{2}\right)}{\ln b} < 1,$$

since

$$\frac{b+1}{2} < b$$

for $b \geq 2$.

3. Example. Suppose $b = 5$. Then $\phi(b) = 4$, $\beta = 2$, and

$$\theta = \frac{\ln 3}{\ln 5} \approx .68$$

and so

$$\begin{aligned} \text{constant } c &\approx 2\left(\frac{5(.32)}{.68(4)}\right)^{.68}\left(\frac{1}{.32}\right) \\ &\approx 2\left(\frac{1.6}{2.72}\right)^{.68}\left(\frac{1}{.32}\right) \\ &\approx 2.18. \end{aligned}$$

Therefore,

$$S(x) \leq 2.18x^{.68}.$$

References

1. H. Gupta, "Powers of 2 and Sums of Distinct Powers of 3," *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.,* 602–633 (1978), 151–158.
2. W. Narkiewicz, "A Note on a Paper of H. Gupta Concerning Powers of Two and Three," *Univ. Beograd Elektrotehn. Fak. Ser. Mat.,* 678–715 (1980), 173–174.

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