BOUNDS ON A REMAINDER TERM ASSOCIATED WITH THE NUMBER OF BASE q DIGITS $\geq$ d FUNCTION

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Abstract

Let $q \geq 2$ be a fixed integer. The base q representation of a positive integer k can be written in the form

$$k = \sum_{r=0}^{\infty} a_r(q,k) q^r, \text{ where } a_r(q,k) \in \{0, 1, \dots, q-1\}.$$

To represent and study the 'number of base q digits $\geq d$' function we need to introduce the bracket notation. The basic idea is to enclose a true-or-false statement in brackets, and to say that the result is 1 if the statement is true, 0 if the statement is false. For example,

$$[p \text{ prime}] = \begin{cases} 1, & \text{if } p \text{ is a prime number;} \\ 0, & \text{if } p \text{ is not a prime number.} \end{cases}$$

Let d be a nonzero base q digit. Define the 'number of base q digits $\geq d$' function as

$$\alpha_{\geq d}(q,k) = \sum_{r=0}^{\infty} [a_r(q,k) \geq d].$$

For an integer $n \geq 1$, let

$$A_{\geq d}(q,n) = \sum_{k=1}^{n-1} \alpha_{\geq d}(q,k).$$

First, we will show that

$$A_{\geq d}(q,n) = \left(1 - \frac{d}{q}\right) n \log_q n + O(n).$$

Second, we define

$$S_{\geq d}(q,n) = A_{\geq d}(q,n) - \left(1 - \frac{d}{q}\right) n \lfloor \log_q n \rfloor$$

where $\log_q$ denote the logarithm function with base q and $\lfloor \cdot \rfloor$ denotes the greatest integer function. We then show that if

$$c > \max\left\{\frac{d}{q}, 1 - \frac{d}{q}\right\},$$

then

$$-c < \frac{S_{\geq d}(q,n)}{n} < 1 - \frac{d}{q}.$$

1. Introduction

Let $q \geq 2$ be a fixed integer. The base q representation of a positive integer k can be written in the form

$$k = \sum_{r=0}^{\infty} a_r(q,k) q^r, \text{ where } a_r(q,k) \in \{0, 1, \ldots, q-1\}.$$

To represent and study the 'number of base q digits $\geq d$' function we need to introduce the bracket notation. The basic idea is to enclose a true-or-false statement in brackets, and to say that the result is 1 if the statement is true, 0 if the statement is false. For example,

$$[p \text{ prime}] = \begin{cases} 1, & \text{if } p \text{ is a prime number;} \\ 0, & \text{if } p \text{ is not a prime number.} \end{cases}$$

Let d be a nonzero base q digit. Define the 'number of base q digits $\geq d$' function as

$$\alpha_{\geq d}(q,k) = \sum_{r=0}^{\infty} [a_r(q,k) \geq d].$$

The 'number of base q digits $\geq d$' function is a generalization of the 'number of large digits' function studied in [1,2].

For an integer $n \geq 1$, let

$$A_{\geq d}(q,n) = \sum_{k=1}^{n-1} \alpha_{\geq d}(q,k).$$

2. Properties of $\mathbf{A_{\geq d}(q,n)}$

We begin by stating and proving some basic properties of the function $A_{\geq d}(q,n)$.

Lemma 1. Let $q \geq 2$ be an integer, s and n be positive integers, and d be nonzero base q digit. Then

$$A_{\geq d}(q,q^s) = \left(1 - \frac{d}{q}\right) sq^s \tag{1}$$

$$\text{and} \quad A_{\geq d}(q,qn) = qA_{\geq d}(q,n) + (q-d)n. \tag{2}$$

Proof. We first prove (1) by induction on s. For $s = 1$ we have

$$A_{\geq d}(q,q^1) = q - d = \left(1 - \frac{d}{q}\right) \cdot 1 \cdot q^1.$$

Now for the induction step we assume that $s \geq 2$ and that (1) is true for $s-1$. Then by counting the number of base q digits $\geq d$ in the numbers from 1 to q^{s-1}, $1 \cdot q^{s-1}$ to $2 \cdot q^{s-1}$, ..., $(q-1) \cdot q^{s-1}$ to $q \cdot q^{s-1}$, we have

$$A_{\geq d}(q,q^s) = \sum_{1 \leq r < q^{s-1}} \alpha_{\geq d}(q,r) + \sum_{t=1}^{q-1} \sum_{tq^{s-1} \leq r < (t+1)q^{s-1}} \alpha_{\geq d}(q,r).$$

Setting $r = tq^{s-1} + u$, where $0 \leq u < q^{s-1}$ in the second (inner) sum and using the fact that $\alpha_{\geq d}(q,r) = [t \geq d] + \alpha_{\geq d}(q,u)$, it follows that

$$\begin{aligned}
A_{\geq d}(q,q^s) &= \sum_{1 \leq r < q^{s-1}} \alpha_{\geq d}(q,r) + \sum_{t=1}^{q-1} \sum_{0 \leq u < q^{s-1}} \Big([t \geq d] + \alpha_{\geq d}(q,u) \Big) \\
&= \sum_{0 \leq u < q^{s-1}} \sum_{t=1}^{q-1} [t \geq d] + \sum_{t=0}^{q-1} \sum_{0 \leq u < q^{s-1}} \alpha_{\geq d}(q,u) \\
&= (q-d)q^{s-1} + qA_{\geq d}(q,q^{s-1}).
\end{aligned}$$

Using the induction hypothesis it follows that

$$A_{\geq d}(q,q^s) = \left(1-\frac{d}{q}\right)q^s + q\left(1-\frac{d}{q}\right)(s-1)q^{s-1} = \left(1-\frac{d}{q}\right)sq^s.$$

Thus, (1) is true for s. By mathematical induction we have proved (1).

To prove (2) we count the number of base q digits $\geq d$ from 1 to $qn-1$ by counting the number of base q digits $\geq d$ in the leading digits from 1 to $qn-1$ and then counting the number of base q digits $\geq d$ in the units digits from 1 to $qn-1$. The first quantity is $qA_{\geq d}(q,n)$ and the second quantity is $(q-d)n$. This proves (2).

Corollary 2. Let $q \geq 2$ be an integer, s be a positive integer, and a and d be nonzero base q digits. Then

$$A_{\geq d}(q,aq^s) = a\left(1-\frac{d}{q}\right)sq^s + (a-d)[a \geq d]q^s. \tag{3}$$

Proof. We will prove this result by counting the number of base q digits $\geq d$ from 1 to aq^s-1 by counting the number of base q digits $\geq d$ in the trailing digits from 1 to aq^s-1 and then counting the number of base q digits $\geq d$ in the leading digit from 1 to aq^s-1. By (2) and the fact that there are a copies of the trailing digits from 1 to q^s-1 in 1 to aq^s-1 the first quantity is

$$a\left(1-\frac{d}{q}\right)sq^s$$

and the number of base q digits $\geq d$ in the leading digits from 1 to aq^s-1 is

$$(a-d)[a \geq d]q^s.$$

The result follows.

3. Main Term and Error Term

The next result gives a main term and an error term for $A_{\geq d}(q,n)$.

Theorem 3. Let $q \geq 2$ be an integer, d be a nonzero digit in base q, and n be a positive integer. Then

$$A_{\geq d}(q,n) = \left(1 - \frac{d}{q}\right) n \log_q n + O(n),$$

where $\log_q$ denotes the logarithm function with base q.

Proof. We begin by observing that if $s \geq 1$ and a is a nonzero base q digit, then the representation of k in base q contains the term aq^s if and only if k can be expressed in the form

$$k = mq^{s+1} + \mu,$$

where $m \geq 0$ and $aq^s \leq \mu < (a+1)q^s$. Hence $f(n,s,a)$, the number of positive integers not exceeding n whose representation in base q contains the term aq^s, is given by

$$\begin{aligned} f(n,s,a) &= \sum_{\substack{mq^{s+1}+\mu \leq n \\ m \geq 0;\ aq^s \leq \mu < (a+1)q^s}} 1 \\ &= \sum_{aq^s \leq \mu < (a+1)q^s} \left(\frac{n}{q^{s+1}} + O(1)\right) = \frac{n}{q} + O(q^s). \end{aligned}$$

But clearly

$$A_{\geq d}(q,n) = \sum_{\substack{0 \leq s \leq \log_q n \\ 0 \leq a \leq q-1}} [d \geq a] f(n,s,a).$$

Therefore, by the formula for $f(n,s,a)$,

$$\begin{aligned} A_{\geq d}(q,n) &= \sum_{\substack{0 \leq s \leq \log_q n \\ 0 \leq a \leq q-1}} [d \geq a] \left(\frac{n}{q} + O(q^s)\right) \\ &= (q-d)\frac{n}{q} \log_q n + O(n) + (q-d) O\left(\frac{1 - q^{\log_q x + 1}}{1-q}\right) \\ &= \left(1 - \frac{d}{q}\right) n \log_q n + O(n). \end{aligned}$$

4. Upper Bound

In [3], Foster derived some upper and lower bounds for the difference between the 'sum of digits' function and its main term. We would like to do the same for the 'number of base q digits $\geq d$' function. Let $q \geq 2$ be an integer, d a nonzero digit in base q, and n a positive integer. Then, we define

$$S_{\geq d}(q,n) = A_{\geq d}(q,n) - \left(1 - \frac{d}{q}\right) n \lfloor \log_q n \rfloor.$$

Here, $\lfloor \cdot \rfloor$ denotes the greatest integer function. We next have the following property of $S_{\geq d}$. This is a corollary to Lemma 1.

Corollary 4. Let k be a nonnegative integer and n be a positive integer. Then

$$\frac{S_{\geq d}(q, q^k n)}{q^k n} = \frac{S_{\geq d}(q,n)}{n}. \tag{4}$$

Proof. The proof is by induction on k. If $k = 0$ the result is clear. Now assume that $k \geq 1$ and that (4) is true for k. Then by the definition of $S_{\geq d}$ and (2)

$$\begin{aligned}
S_{\geq d}(q, q^{k+1}n) &= A_{\geq d}(q, q^{k+1}n) - \left(1 - \frac{d}{q}\right) q^{k+1} n \lfloor \log_q q^{k+1} n \rfloor \\
&= qA_{\geq d}(q, q^k n) + (q-d)q^k n - \left(1 - \frac{d}{q}\right) q^{k+1} n \left(1 + \lfloor \log_q q^k n \rfloor\right) \\
&= qA_{\geq d}(q, q^k n) + \left(1 - \frac{d}{q}\right) q^{k+1} n - \left(1 - \frac{d}{q}\right) q^{k+1} n - q\left(1 - \frac{d}{q}\right) q^k n \lfloor \log_q q^k n \rfloor \\
&= qA_{\geq d}(q, q^k n) - q\left(1 - \frac{d}{q}\right) q^k n \lfloor \log_q q^k n \rfloor \\
&= qS_{\geq d}(q, q^k n).
\end{aligned}$$

By the induction hypothesis,

$$\frac{S_{\geq d}(q, q^{k+1}n)}{q^{k+1}n} = \frac{qS_{\geq d}(q, q^k n)}{q^{k+1}n} = \frac{S_{\geq d}(q, q^k n)}{q^k n} = \frac{S_{\geq d}(q,n)}{n}.$$

Therefore, (4) is true for $k+1$. Thus, by mathematical induction the result is true.

Because of (4), to find bounds on

$$\frac{S_{\geq d}(q,n)}{n}$$

we can assume without loss of generality that $n \not\equiv 0 \pmod{q}$. This leads to the following definition.

Definition 5. Let $n \not\equiv 0 \pmod{q}$. Then n is of the form $n = n_m$ where

$$n_m = a_0 q^{t_0} + a_1 q^{t_0+t_1} + a_2 q^{t_0+t_1+t_2} + \cdots + a_m q^{t_0+t_1+t_2+\cdots+t_m}$$

for some nonnegative integer m, $t_0 = 0$, positive integers $t_1, t_2, \ldots, t_m$ and nonzero coefficients $a_0, a_1, a_2, \ldots, a_m \in \{1, 2, \ldots, q-1\}$. Also, define

$$n_0 = a_0 \text{ and } n_i = a_0 + a_1 q^{t_1} + \cdots + a_i q^{t_1+\cdots+t_i}$$

for $1 \leq i \leq m$.

Lemma 6. Let $q \geq 2$ and let $m \geq 1$. Then

$$\begin{aligned} A_{\geq d}(q,n_m) &= A_{\geq d}(q,n_{m-1}) + a_m\left(1-\frac{d}{q}\right)(t_1+\cdots+t_m)q^{t_1+\cdots+t_m} \\ &+ [a_m \geq d]n_{m-1} + (a_m - d)[a_m \geq d]q^{t_1+\cdots+t_m} \qquad (5) \end{aligned}$$

and

$$\begin{aligned} S_{\geq d}(q,n_m) &= S_{\geq d}(q,n_{m-1}) + (a_m - d)[a_m \geq d]q^{t_1+\cdots+t_m} \\ &+ \left([a_m \geq d] - \left(1-\frac{d}{q}\right)t_m\right)n_{m-1} \qquad (6) \end{aligned}$$

Proof. By (3) we have that

$$\begin{aligned} A_{\geq d}(q,n_m) &= A_{\geq d}(q,a_m q^{t_1+\cdots+t_m}) + \sum_{a_m q^{t_1+\cdots+t_m} \leq r < n_m} \alpha_{\geq d}(q,r) \\ &= a_m\left(1-\frac{d}{q}\right)(t_1+\cdots+t_m)q^{t_1+\cdots+t_m} + (a_m - d)[a_m \geq d]q^{t_1+\cdots+t_m} \\ &+ [a_m \geq d]n_{m-1} + A_{\geq d}(q,n_{m-1}). \end{aligned}$$

This is (5). Using the definition of $S_{\geq d}$ and (5), we have

$$
\begin{aligned}
&S_{\geq d}(q, n_m) = A_{\geq d}(q, n_m) - \left(1 - \frac{d}{q}\right) n_m \lfloor \log_q n_m \rfloor \\
&= A_{\geq d}(q, n_{m-1}) + a_m \left(1 - \frac{d}{q}\right)(t_1 + \cdots + t_m) q^{t_1 + \cdots + t_m} \\
&+ [a_m \geq d] n_{m-1} + (a_m - d)[a_m \geq d] q^{t_1 + \cdots + t_m} - \left(1 - \frac{d}{q}\right) n_m \lfloor \log_q n_m \rfloor \\
&= A_{\geq d}(q, n_{m-1}) + a_m \left(1 - \frac{d}{q}\right)(t_1 + \cdots + t_m) q^{t_1 + \cdots + t_m} \\
&+ [a_m \geq d] n_{m-1} + (a_m - d)[a_m \geq d] q^{t_1 + \cdots + t_m} \\
&- \left(1 - \frac{d}{q}\right)(a_m q^{t_1 + \cdots + t_m} + n_{m-1}) \left(t_m + \lfloor \log_q n_{m-1} \rfloor \right) \\
&= A_{\geq d}(q, n_{m-1}) + a_m \left(1 - \frac{d}{q}\right)(t_1 + \cdots + t_m) q^{t_1 + \cdots + t_m} + [a_m \geq d] n_{m-1} \\
&+ (a_m - d)[a_m \geq d] q^{t_1 + \cdots + t_m} - \left(1 - \frac{d}{q}\right) n_{m-1} \lfloor \log_q n_{m-1} \rfloor - \left(1 - \frac{d}{q}\right)(a_m t_m q^{t_1 + \cdots + t_m}) \\
&- \left(1 - \frac{d}{q}\right) a_m q^{t_1 + \cdots + t_m} (t_1 + \cdots t_{m-1}) - \left(1 - \frac{d}{q}\right) n_{m-1} t_m \\
&= A_{\geq d}(q, n_{m-1}) - \left(1 - \frac{d}{q}\right) n_{m-1} \lfloor \log_q n_{m-1} \rfloor + [a_m \geq d] n_{m-1} \\
&+ (a_m - d)[a_m \geq d] q^{t_1 + \cdots + t_m} - \left(1 - \frac{d}{q}\right) n_{m-1} t_m \\
&= S_{\geq d}(q, n_{m-1}) + [a_m \geq d] n_{m-1} \\
&+ (a_m - d)[a_m \geq d] q^{t_1 + \cdots + t_m} - \left(1 - \frac{d}{q}\right) n_{m-1} t_m
\end{aligned}
$$

This is (6).

Theorem 7. Let $q \geq 2$, $1 \leq d \leq q - 1$ and $m \geq 0$. Then

$$\frac{S_{\geq d}(q, n_m)}{n_m} < 1 - \frac{d}{q}.$$

Proof. The proof is by induction on m. If $m = 0$, we have $n_m = n_0 = a_0$ so that

$$\frac{S_{\geq d}(q, n_0)}{n_0} = \frac{(a_0 - d)[a_0 \geq d]}{a_0} < 1 - \frac{d}{q},$$

which is the stated result.

Now choose $m \geq 1$ and assume that

$$\frac{S_{\geq d}(q, n_{m-1})}{n_{m-1}} < 1 - \frac{d}{q}.$$

By (6) and since $t_m \geq 1$, we have that

$$\begin{aligned}
&S_{\geq d}(q, n_m) = S_{\geq d}(q, n_{m-1}) + (a_m - d)[a_m \geq d]q^{t_1+\cdots+t_m} \\
&+ \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)t_m\right)n_{m-1} \\
&= S_{\geq d}(q, n_{m-1}) + (a_m q^{t_1+\cdots+t_m} - dq^{t_1+\cdots+t_m})[a_m \geq d] \\
&+ \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)t_m\right)n_{m-1} \\
&= S_{\geq d}(q, n_{m-1}) + (n_m - n_{m-1} - dq^{t_1+\cdots+t_m})[a_m \geq d] \\
&+ \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)t_m\right)n_{m-1} \\
&\leq S_{\geq d}(q, n_{m-1}) + (n_m - n_{m-1} - dq^{t_1+\cdots+t_m})[a_m \geq d] \\
&+ \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)\right)n_{m-1} \\
&= S_{\geq d}(q, n_{m-1}) + (n_m - dq^{t_1+\cdots+t_m})[a_m \geq d] \\
&- \left(1 - \frac{d}{q}\right)n_{m-1}
\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
&S_{\geq d}(q, n_{m-1}) + (n_m - dq^{t_1+\cdots+t_m})[a_m \geq d] - \left(1 - \frac{d}{q}\right)n_{m-1} \\
&< \left(1 - \frac{d}{q}\right)n_{m-1} + (n_m - dq^{t_1+\cdots+t_m})[a_m \geq d] - \left(1 - \frac{d}{q}\right)n_{m-1} \\
&= \left(n_m - dq^{t_1+\cdots+t_m}\right)[a_m \geq d].
\end{aligned}$$

Therefore,

$$\frac{S_{\geq d}(q, n_m)}{n_m} < \left(1 - \frac{dq^{t_1+\cdots+t_m}}{n_m}\right)[a_m \geq d].$$

But, $n_m < q^{t_1+\cdots+t_m+1}$. Thus,

$$\frac{S_{\geq d}(q, n_m)}{n_m} < 1 - \frac{d}{q}.$$

Therefore, by mathematical induction, the result is true.

5. Lower Bound

The following lemma will help in finding a lower bound for

$$\frac{S_{\geq d}(q, n_m)}{n_m}.$$

Lemma 8. Let $m \geq 1$ be an integer. Then

$$\frac{n_{m-1}}{n_m} < \frac{1}{1 + a_m q^{t_m - 1}}. \tag{7}$$

Proof. See [3, p. 113].

Theorem 9. Let $m \geq 0$, $q \geq 2$, and $1 \leq d \leq q - 1$. Also, let

$$c > \max\left\{\frac{d}{q}, 1 - \frac{d}{q}\right\}.$$

Then

$$\frac{S_{\geq d}(q, n_m)}{n_m} > -c.$$

Proof. By induction on m. Let $m = 0$. Then

$$\frac{S_{\geq d}(q, n_0)}{n_0} = \frac{(a_0 - d)[a_0 \geq d]}{a_0} \geq 0 > -c.$$

For the induction step, let $m \geq 1$ and assume that

$$\frac{S_{\geq d}(q, n_{m'})}{n_{m'}} > -c \quad \text{for all integers} \quad m' \quad \text{with} \quad 0 \leq m' \leq m - 1.$$

By (6), the induction hypothesis, and the fact that $(a_m - d)[a_m \geq d]q^{t_1 + \cdots + t_m} \geq 0$, we have that

$$\begin{aligned} &S_{\geq d}(q, n_m) \\ &= S_{\geq d}(q, n_{m-1}) + (a_m - d)[a_m \geq d]q^{t_1 + \cdots + t_m} + \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)t_m\right)n_{m-1} \\ &> -c\dot{n}_{m-1} + \left([a_m \geq d] - \left(1 - \frac{d}{q}\right)t_m\right)n_{m-1}. \end{aligned}$$

Thus,

$$\frac{S_{\geq d}(q,n_m)}{n_m} > \frac{n_{m-1}}{n_m}\left([a_m \geq d] - \left(1-\frac{d}{q}\right)t_m - c\right) > -c$$

provided that

$$\frac{n_{m-1}}{n_m} < \frac{c}{c-[a_m \geq d] + \left(1-\frac{d}{q}\right)t_m}.$$

To prove this last inequality we are assuming that

$$[a_m \geq d] < \left(1-\frac{d}{q}\right)t_m + c$$

or

$$c > [a_m \geq d] - \left(1-\frac{d}{q}\right)t_m.$$

This is true for all $1 \leq a_m \leq q-1$, $t_m \geq 1$, d a nonzero digit in base $q \geq 2$ since $c > d/q$. By (7),

$$\frac{n_{m-1}}{n_m} < \frac{1}{1+a_m q^{t_m-1}},$$

so if we can show that

$$\frac{1}{1+a_m q^{t_m-1}} < \frac{c}{c-[a_m \geq d] + (1-\frac{d}{q})t_m}$$

for all $a_m \geq 1$ and $t_m \geq 1$ we are done. But this last inequality is true if and only if

$$c-[a_m \geq d] + \left(1-\frac{d}{q}\right)t_m < c + a_m c q^{t_m-1}.$$

But this inequality is true if and only if

$$c > \frac{(1-\frac{d}{q})t_m - [a_m \geq d]}{a_m q^{t_m-1}}.$$

But this last inequality is true for all $t_m \geq 1$, $a_m \geq 1$, and d a nonzero digit in base $q \geq 2$ since $t_m/q^{t_m-1} \leq 1$ and $c > 1-(d/q)$ so we are done. Therefore, the result is true for m. Hence, by mathematical induction, the result is true.

6. Questions

In [3], Foster's 'sum of base 2 digits' function is identical to our 'number of base 2 digits ≥ 1' function. Translating his result to our situation gives us the following definition and theorem. The proof of the result follows Foster's proof.

Definition 11. Let m be a nonnegative integer. Let

$$h_2(m) = \frac{2^{2m} - 1}{13 \cdot 2^{2m} - 1}.$$

Theorem 12. Let m be a nonnegative integer. Then

$$-h_2(m) \leq \frac{S_{\geq 1}(2, n_m)}{n_m} \leq \frac{1}{2} - \frac{m+1}{2(2^{m+1} - 1)}.$$

For base 2 numbers with units digit 1, the largest value of

$$\frac{S_{\geq 1}(2, n_m)}{n_m}$$

occurs when $n_m = 1 + 2 + 2^2 + \cdots + 2^m$ and the smallest value occurs $n_m = 1 + 2^2 + 2^4 + \cdots + 2^{2(m-1)} + 2^{2(m+1)}$ for $m \geq 1$.

Are there similar results for other bases and digits? For example, what about $q = 3$ and $d = 1$. We conclude with the following definition and conjecture.

Definition 13. Let

$$h_3(m) = \begin{cases} \frac{11 \cdot 27^{\frac{m+1}{2}} - 63}{246 \cdot 27^{\frac{m+1}{2}} - 90} & \text{if } m \text{ is odd} \\ \frac{11 \cdot 27^{\frac{m}{2}} - 11}{246 \cdot 27^{\frac{m}{2}} - 12} & \text{if } m \text{ is even.} \end{cases}$$

Also, let

$$g_3(m) = \frac{2 \cdot 3^{m+1} - m - 3}{3 \cdot 3^{m+1} - 3}.$$

Conjecture 14. Let $m \geq 0$. Then

$$-h_3(m) \leq \frac{S_{\geq 1}(3, n_m)}{n_m} \leq g_3(m).$$

Equality on the left occurs for $n_0 = 1$, $n_1 = 28 = 1 + 3^3$, and for $m \geq 2$,

$$n_m = \begin{cases} \sum_{i=0}^{\frac{m-2}{2}} \left(3^{i(1+2)} + 3^{i(1+2)+1}\right) + 3^{3\frac{m}{2}+1} = \frac{41}{13} 27^{\frac{m}{2}} - \frac{2}{13} & m \text{ even} \\ \sum_{i=0}^{\frac{m-3}{2}} \left(3^{i(2+1)} + 3^{i(2+1)+2}\right) + 3^{3\frac{m+1}{2}} + 3^{3\frac{m+3}{2}} = \frac{41}{39} 27^{\frac{m+1}{2}} - \frac{5}{13} & m \text{ odd.} \end{cases}$$

Equality on the right occurs for $n_m = 3^{m+1} - 1$.

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