AN IDENTITY FOR PERIOD k SECOND ORDER LINEAR RECURRENCE SYSTEMS

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Abstract. Let a_1, a_2, b_1, and b_2 be real numbers. The period 2 second order linear recurrence system is defined to be the sequence $x_0 = 1$, $x_1 = a_1$, and

$$\begin{aligned} x_{2n+2} &= a_2x_{2n+1} + b_1x_{2n}, \\ x_{2n+3} &= a_1x_{2n+2} + b_2x_{2n+1}, \end{aligned}$$

for $n \geq 0$. We will show that for $n \geq 4$,

$$x_n = (a_1a_2 + b_1 + b_2)x_{n-2} - b_1b_2x_{n-4}.$$

Then we will then find and prove a similar identity for the period $k \geq 2$ second order linear recurrence system.

1. Introduction

Let a_1 and b_1 be real numbers. A second order linear recurrence is defined by $x_0 = 1$, $x_1 = a_1$ and for $n \geq 2$,

$$x_n = a_1x_{n-1} + b_1x_{n-2}.$$

We define this to be the period 1 second order linear recurrence system. We now define the period 2 second order linear recurrence system [2].

Definition 1.1. *Let a_1, a_2, b_1, and b_2 be real numbers. The period 2 second order linear recurrence system is defined to be the sequence $x_0 = 1$, $x_1 = a_1$, and*

$$\begin{aligned} x_{2n+2} &= a_2x_{2n+1} + b_1x_{2n}, \\ x_{2n+3} &= a_1x_{2n+2} + b_2x_{2n+1}, \end{aligned}$$

for $n \geq 0$.

Next, we define a period $k \geq 2$ second order linear recurrence system [2].

Definition 1.2. *Let $k \geq 2$ be a positive integer. Let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be real numbers. The period k second order linear recurrence* *system is defined to be the sequence* $x_0 = 1$, $x_1 = a_1$, *and*

$$\begin{aligned} x_{kn+2} &= a_2x_{kn+1} + b_1x_{kn}, \\ x_{kn+3} &= a_3x_{kn+2} + b_2x_{kn+1}, \\ \vdots &= \vdots \\ x_{kn+k} &= a_kx_{kn+k-1} + b_{k-1}x_{kn+k-2}, \\ x_{kn+k+1} &= a_1x_{kn+k} + b_kx_{kn+k-1}, \end{aligned}$$

for $n \geq 0$.

2. Period 2 and Period 3 Identities

We will state and prove the identity for the period 2 second order linear recurrence system.

Theorem 2.1. *Let* $\{x_n\}$ *be the period* 2 *second order linear recurrence system. Then for* $n \geq 4$,

$$x_n = (a_1a_2 + b_1 + b_2)x_{n-2} - b_1b_2x_{n-4}.$$

Proof. We will consider two cases.

Case 1: $n \geq 4$ is even. Then

$$\begin{aligned} x_n &= a_2x_{n-1} + b_1x_{n-2} \\ &= a_2(a_1x_{n-2} + b_2x_{n-3}) + b_1x_{n-2} \\ &= (a_1a_2 + b_1)x_{n-2} + b_2(a_2x_{n-3}) \\ &= (a_1a_2 + b_1)x_{n-2} + b_2(x_{n-2} - b_1x_{n-4}) \\ &= (a_1a_2 + b_1 + b_2)x_{n-2} - b_1b_2x_{n-4}. \end{aligned}$$

Case 2: $n \geq 4$ is odd. Then

$$\begin{aligned} x_n &= a_1x_{n-1} + b_2x_{n-2} \\ &= a_1(a_2x_{n-2} + b_1x_{n-3}) + b_2x_{n-2} \\ &= (a_1a_2 + b_2)x_{n-2} + b_1(a_1x_{n-3}) \\ &= (a_1a_2 + b_2)x_{n-2} + b_1(x_{n-2} - b_2x_{n-4}) \\ &= (a_1a_2 + b_1 + b_2)x_{n-2} - b_1b_2x_{n-4}. \end{aligned}$$

This completes the proof. □

Next, we give the identity and proof for the period 3 second order linear recurrence system.

Theorem 2.2. *Let* $\{x_n\}$ *be the period* 3 *second order linear recurrence system. Then for* $n \geq 6$,

$$x_n = (a_1a_2a_3 + a_1b_2 + a_2b_3 + a_3b_1)x_{n-3} + b_1b_2b_3x_{n-6}.$$

Proof. We will prove the first of three cases.

Case 1: $n \geq 6$ and $n \equiv 0 \pmod{3}$. Then

$$x_n = a_3 x_{n-1} + b_2 x_{n-2} = a_3(a_2 x_{n-2} + b_1 x_{n-3}) + b_2 x_{n-2}$$
$$= a_2 a_3 x_{n-2} + b_2 x_{n-2} + a_3 b_1 x_{n-3}$$
$$= a_2 a_3 (a_1 x_{n-3} + b_3 x_{n-4}) + b_2(a_1 x_{n-3} + b_3 x_{n-4}) + a_3 b_1 x_{n-3}$$
$$= (a_1 a_2 a_3 + a_1 b_2 + a_3 b_1) x_{n-3} + a_2 b_3 (a_3 x_{n-4}) + b_2 b_3 x_{n-4}$$
$$= (a_1 a_2 a_3 + a_1 b_2 + a_3 b_1) x_{n-3} + a_2 b_3 (x_{n-3} - b_2 x_{n-5}) + b_2 b_3 x_{n-4}$$
$$= (a_1 a_2 a_3 + a_1 b_2 + a_2 b_3 + a_3 b_1) x_{n-3} - a_2 b_2 b_3 x_{n-5}$$
$$+ b_2 b_3 (a_2 x_{n-5} + b_1 x_{n-6})$$
$$= (a_1 a_2 a_3 + a_1 b_2 + a_2 b_3 + a_3 b_1) x_{n-3} + b_1 b_2 b_3 x_{n-6}.$$

The proofs of the other two cases are similar. □

3. Description of the Period k Identity

Next we state the identities for periods 4, 5, and 6. We will write the subscripts of the terms in descending order. This will parallel how the proof of the identity is derived.

Period 4, 5, and 6 Identities.

(1) Period 4.

$$x_n = (a_4 a_3 a_2 a_1 + a_4 a_3 b_1 + a_4 b_2 a_1 + b_4 a_3 a_2 + b_3 a_2 a_1 + b_4 b_2$$
$$+ b_3 b_1) x_{n-4} - b_1 b_2 b_3 b_4 x_{n-8}.$$

(2) Period 5.

$$x_n = (a_5 a_4 a_3 a_2 a_1 + a_5 a_4 a_3 b_1 + a_5 a_4 b_2 a_1 + a_5 b_3 a_2 a_1 + b_5 a_4 a_3 a_2$$
$$+ b_4 a_3 a_2 a_1 + a_5 b_3 b_1 + b_5 a_4 b_2 + b_4 a_3 b_1 + b_5 b_3 a_2 + b_4 b_2 a_1) x_{n-5}$$
$$+ b_1 b_2 b_3 b_4 b_5 x_{n-10}.$$

(3) Period 6.

$$x_n = (a_6 a_5 a_4 a_3 a_2 a_1 + a_6 a_5 a_4 a_3 b_1 + a_6 a_5 a_4 b_2 a_1 + a_6 a_5 b_3 a_2 a_1$$
$$+ a_6 b_4 a_3 a_2 a_1 + b_6 a_5 a_4 a_3 a_2 + b_5 a_4 a_3 a_2 a_1 + a_6 a_5 b_3 b_1 + a_6 b_4 a_3 b_1$$
$$+ b_6 a_5 a_4 b_2 + b_5 a_4 a_3 b_1 + a_6 b_4 b_2 a_1 + b_6 a_5 b_3 a_2 + b_5 a_4 b_2 a_1$$
$$+ b_6 b_4 a_3 a_2 + b_5 b_3 a_2 a_1 + b_6 b_4 b_2 + b_5 b_3 b_1) x_{n-6} - b_1 b_2 b_3 b_4 b_5 b_6 x_{n-12}.$$

Bracelet Notation. To construct the identity for x_n in terms of x_{n-k} and x_{n-2k} in the period k case, we need the Lucas numbers. The Lucas numbers, 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. It can be shown that for $n \geq 1$, L_n counts the number of ways to create a bracelet of length n using beads of length one or two [1]. To write the different bracelets of length k, using beads of length one and two, we begin by numbering the bracelet positions from 1 to k. To specify the type of bead and its position in the bracelet, we let the letter a denote a bead of length one and the letter b denote a bead of length two. Furthermore, if a bead of length one occupies position i, the subscript on the a will be i. If a bead of length two occupies positions i and $i+1$, the subscript on the b will be i. However, if a bead of length two occupies positions k and 1, the subscript on the b will be k. To denote the beads and their positions in a bracelet, we list the a's and b's with their subscripts in the bracelet. The order of the beads is not important, although we will write the subscripts in decreasing order. For example,

$$a_6 a_5 b_3 a_2 a_1$$

denotes a bracelet of length 6 with beads of length one at positions 1, 2, 5, and 6 and a bead of length two at positions 3 and 4. Also,

$$b_7 b_5 a_4 a_3 a_2$$

denotes a bracelet of length 7 with beads of length one at positions 2, 3, and 4 and beads of length two at positions 5 and 6 and positions 7 and 1.

Our identity for x_n, where x_n is a period k second order linear recurrence system, is a constant times x_{n-k} plus a constant times x_{n-2k}. The constants (coefficients of x_{n-k} and x_{n-2k}) are given next.

<u>Coefficient of x_{n-k}</u>.

(1) Write all possible terms of a's and b's where the number of a's plus twice the number of b's is k.
(2) If a term begins with an a, label the a subscript with a k. Reduce the subscript of the next bead by 1 if the next symbol is an a, otherwise reduce the subscript by 2.
(3) If a term begins with a b, label the b subscript with either a k or $k-1$. Reduce the subscript of the next bead by 1 if the next symbol is an a, otherwise reduce the subscript by 2.
(4) Continue labeling the subscripts of the a's and b's, reducing the subscript of the next bead by 1 if the next symbol is an a, otherwise reducing the subscript by 2.
(5) The coefficient of x_{n-k} is the sum of all these terms.
(6) Since this labelling completely enumerates the number of ways to create a bracelet of length k, the number of terms with an x_{n-k} is L_k, the kth Lucas number.

<u>Coefficient of x_{n-2k}</u>. The coefficient is $(-1)^{k-1} b_1 b_2 \cdots b_k$.

<u>Identity for Period 7</u>. We will construct the identity for the period 7 second order linear recurrence system. The term x_n will be equal to some

coefficient times x_{n-7} plus some coefficient times x_{n-14}. To obtain the coefficient of x_{n-7} list the configurations of a's and b's where the number of a's plus twice the number of b's is 7 is

$aaaaaaa$,

$aaaaab$, $aaaaba$, $aaabaa$, $aabaaa$, $abaaaa$, $baaaaa$,

$aaabb$, $aabab$, $abaab$, $baaab$, $aabba$, $ababa$, $baaba$, $abbaa$, $babaa$, $bbaaa$,

$abbb$, $babb$, $bbab$, $bbba$.

Next, label each configurations according to the rules for x_{n-7}.

$a_7a_6a_5a_4a_3a_2a_1,$

$a_7a_6a_5a_4a_3b_1,\ a_7a_6a_5a_4b_2a_1,\ a_7a_6a_5b_3a_2a_1,\ a_7a_6b_4a_3a_2a_1,$

$a_7b_5a_4a_3a_2a_1,\ b_7a_6a_5a_4a_3a_2,\ b_6a_5a_4a_3a_2a_1,$

$a_7a_6a_5b_3b_1,\ a_7a_6b_4a_3b_1,\ a_7b_5a_4a_3b_1,\ b_7a_6a_5a_4b_2,\ b_6a_5a_4a_3b_1,$

$a_7a_6b_4b_2a_1,\ a_7b_5a_4b_2a_1,\ b_7a_6a_5b_3a_2,\ b_6a_5a_4b_2a_1,\ a_7b_5b_3a_2a_1,$

$b_7a_6b_4a_3a_2,\ b_6a_5b_3a_2a_1,\ b_7b_5a_4a_3a_2,\ b_6b_4a_3a_2a_1,$

$a_7b_5b_3b_1,\ b_7a_6b_4b_2,\ b_6a_5b_3b_1,\ b_7b_5a_4b_2,\ b_6b_4a_3b_1,\ b_7b_5b_3a_2,\ b_6b_4b_2a_1.$

The sum of these $L_7 = 29$ terms is the coefficient of x_{n-7}. And, the coefficient of x_{n-14} is $b_1b_2b_3b_4b_5b_6b_7$.

4. Proof of the Period k Identity

Proof. Let $k \geq 2$ be a positive integer. The proofs we gave for the $k = 2$ and $k = 3$ identities involved 2 and 3 cases, respectively. Our proof for the period k case will consist of k cases. The first case will be where the subscript of x_n is a multiple of k. Thus, we will prove the identity for x_{kn} in terms of x_{kn-k} and x_{kn-2k}. We will then explain why the other $k-1$ cases produce similar identities. This will complete our proof.

Write x_{kn} in terms of x_{kn-1} and x_{kn-2} using the defining recurrence relation

$$x_{kn} = a_k x_{kn-1} + b_{k-1} x_{kn-2}. \tag{1}$$

We can write this equality as a tree [3, pp. 64–65]. The root of the tree is x_{kn}. The left child of the root is x_{kn-1} and the right child of the root is x_{kn-2}. The edge from the root to the left child is labeled a_k and the edge from the root to the right child is labelled b_{k-1}. This tree is displayed in Figure 1.

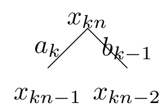

Figure 1.

We can view the tree in Figure 1 as equation (1) in the following way. The root of this tree is the left-hand side of the equation. Each leaf in the tree corresponds to a term in the sum on the right-hand side of the equation. Each term consists of a leaf of the tree times the product of all the labelled branches leading from the leaf to the root of the tree.

Now, consider all the leaves in the tree in Figure 1 labelled x_{kn-1}. We wish to substitute the value of x_{kn-1} given by the recurrence relation in equation (2)

$$x_{kn-1} = a_{k-1}x_{kn-2} + b_{k-2}x_{kn-3}. \tag{2}$$

into equation (1). Equation (2) can be represented by the tree in Figure 2.

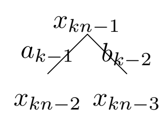

Figure 2.

We now replace all leaves labeled x_{kn-1} in Figure 1 by the tree in Figure 2. The result is the tree given in Figure 3. Again, the root of this tree is equal to the sum of each leaf times the product of the labels of all the branches on the path from the leaf to the root.

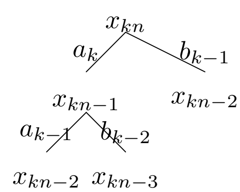

Figure 3.

Next, we consider all leaves in the tree in Figure 3 labeled x_{kn-2}. Replace all leaves x_{kn-2} in this tree by the tree corresponding to the recurrence relation

$$x_{kn-2} = a_{k-2}x_{kn-3} + b_{k-3}x_{kn-4}.$$

The result is the tree in Figure 4.

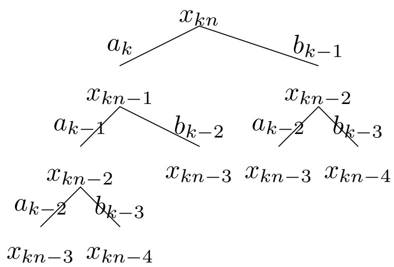

Figure 4.

Continue this process until we have the tree with leaves consisting of x_{kn-k} and x_{kn-k-1}. At this point, the equation associated with the tree has terms containing a's and b's. Each term contains an a_k or b_{k-1} and the a's or b's in each product decrease by 1 or 2 depending on whether or not the next factor in the term is an a or b, respectively. For example, the tree resulting from this process for the case $k = 5$ is given in Figure 5.

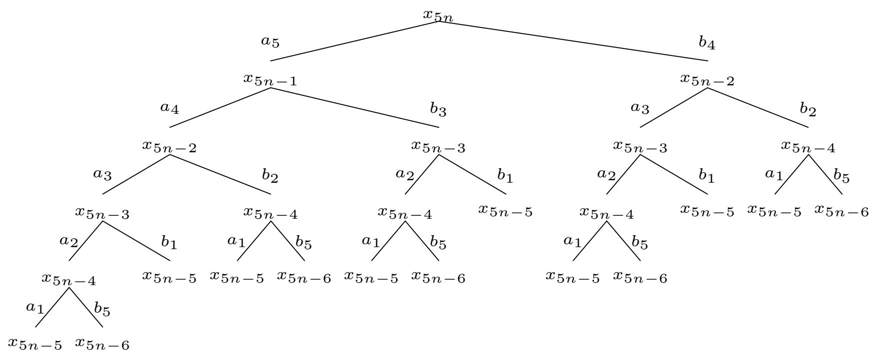

Figure 5.

Next, replace every leaf labeled x_{kn-k-1} in the left subtree of the tree (these terms in our equation would contain both a factor of a_k and b_k) by the tree corresponding to the equation

$$x_{kn-k-1} = \frac{1}{a_k} x_{kn-k} - \frac{b_{k-1}}{a_k} x_{kn-k-2}.$$

For the case $k = 5$, this results in the tree given in Figure 6.

Note that the branches $\frac{1}{a_k}$ and $-\frac{b_{k-1}}{a_k}$ involve the removal of the factor a_k from the leaf's term. This can be done regardless of whether or not $a_k = 0$. Now, the current identity for x_{kn} is in terms of x_{kn-k}, x_{kn-k-1}, and x_{kn-k-2}. The coefficients of x_{kn-k} have terms containing a's and b's, starting with a_k, b_k, or b_{k-1}, where the sequences decrease by 1 or 2 depending on whether or not the next factor in the term is an a or b, respectively. In addition, the number of terms of x_{kn-k} is L_k since these would correspond to exactly the valid bracelets of size k. Therefore, we have reduced x_{kn} to the sum of terms we described involving x_{kn-k} and terms involving x_{kn-k-1} and x_{kn-k-2}. Now, we need to simplify the remaining terms so that we can construct the final term x_{kn-2k} and cancel the other terms. To do this we continue replacing leaves labeled x_{kn-k-1}, x_{kn-k-2}, etc. by trees representing the recurrence relations

$$\begin{aligned} x_{kn-k-1} &= a_{k-1}x_{kn-k-2} + b_{k-2}x_{kn-k-3}, \\ x_{kn-k-2} &= a_{k-2}x_{kn-k-3} + b_{k-3}x_{kn-k-4}, \\ \vdots &= \vdots \end{aligned}$$

We stop the replacement the instant that a factor of a and b with the same subscript appears in the path from a leaf to the root or until we reach x_{kn-2k}. We give the example of this situation when $k = 5$ in Figure 7.

We first show that there is one term of the form $(-1)^{k-1}b_1b_2b_3\cdots b_kx_{kn-2k}$. If k is even, this term comes from the left subtree where we first take a left branch and follow this with k right branches. The factor of a_k is removed by the right branch labelled $-\frac{b_{k-1}}{a_k}$. The term will be negative and contain every b. And if k is odd, the x_{kn-2k} term comes from the right subtree where we take k right branches. The k right branches result in all the b factors. The term will be positive.

Now, all the other terms cancel. To see this, consider a term of x_{kn-k-i}, $1 \le i < k$, from the left subtree which has a repeated subscript of a and b in the path from its leaf to the root. The coefficient will have a negative sign. It will also have a b_k and a b_{k-1} term in it. Starting with the root and writing the factors in order as we go down to the leaf we have

$$a_k = c_{i_1} \cdot c_{i_2} \cdot \cdots \cdot c_{i_n} \cdots c_{i_r} \cdot b_k(-\frac{b_{k-1}}{a_k}) = c_{j_1} \cdot \cdots \cdot c_{j_s}.$$

Here, $i_n = j_s$. We will describe the corresponding term of x_{kn-k-i} with the same coefficient and opposite sign from the right subtree using the left subtree's factors above. Begin with the b_{k-1} factor, mimic the path from the $-\frac{b_{k-1}}{a_k}$ branch to the leaf, continue after the occurrence of the repeated subscript factor's term to the b_k factor, and finally continue from the second branch after the root to the repeated term. This coefficient will be positive. This will be denoted by

$$b_{k-1}c_{j_2}\cdots c_{j_s}c_{i_{n+1}}\cdots c_{i_r}b_kc_{i_2}\cdots c_{i_n}.$$

And we could give a similar description starting with a term of the form x_{kn-k-i}, $1 \le i < k$ from the right subtree and match exactly one term from the left subtree. Therefore, the left subtree term will cancel the right subtree term.

For example, consider the term x_{5n-9} in the left and right subtree of Figure 7. The term from the left subtree is

$$a_5 b_3 a_2 b_5 (-\frac{b_4}{a_5}) b_2$$

and the corresponding term from the right subtree is

$$b_4 b_2 b_5 b_3 a_2.$$

This completes the proof of the first case of the identity. But, the other $k-1$ identities, i.e., x_{kn+1}, x_{kn+2}, ..., x_{kn+k-1} will be proved in the same way as the proof of the identity for x_{kn} and would result in the same identity.

This completes the proof. $\square$

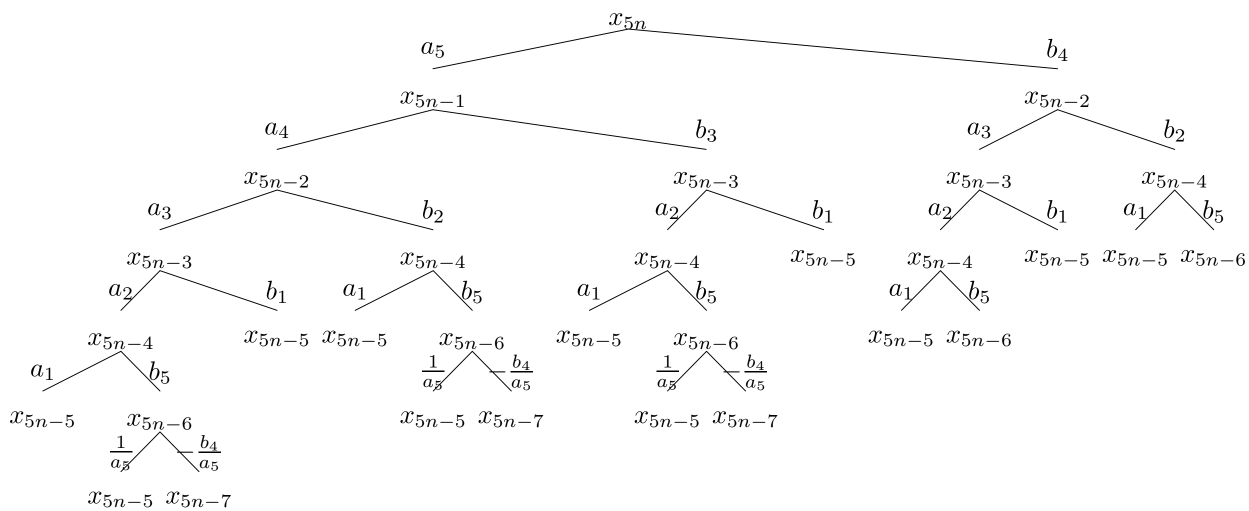
x_{5n}
a_5 b_4
x_{5n-1} x_{5n-2}
a_4 b_3 a_3 b_2
x_{5n-2} x_{5n-3} x_{5n-3} x_{5n-4}
a_3 b_2 a_2 b_1 a_2 b_1 a_1 b_5
x_{5n-3} x_{5n-4} x_{5n-4} x_{5n-5} x_{5n-4} x_{5n-5} x_{5n-5} x_{5n-6}
a_2 b_1 a_1 b_5 a_1 b_5 a_1 b_5
x_{5n-4} x_{5n-5} x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-6}
a_1 b_5 $\frac{1}{a_5}$ $-\frac{b_4}{a_5}$ $\frac{1}{a_5}$ $-\frac{b_4}{a_5}$
x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-7} x_{5n-5} x_{5n-7}
$\frac{1}{a_5}$ $-\frac{b_4}{a_5}$
x_{5n-5} x_{5n-7}

Figure 6.

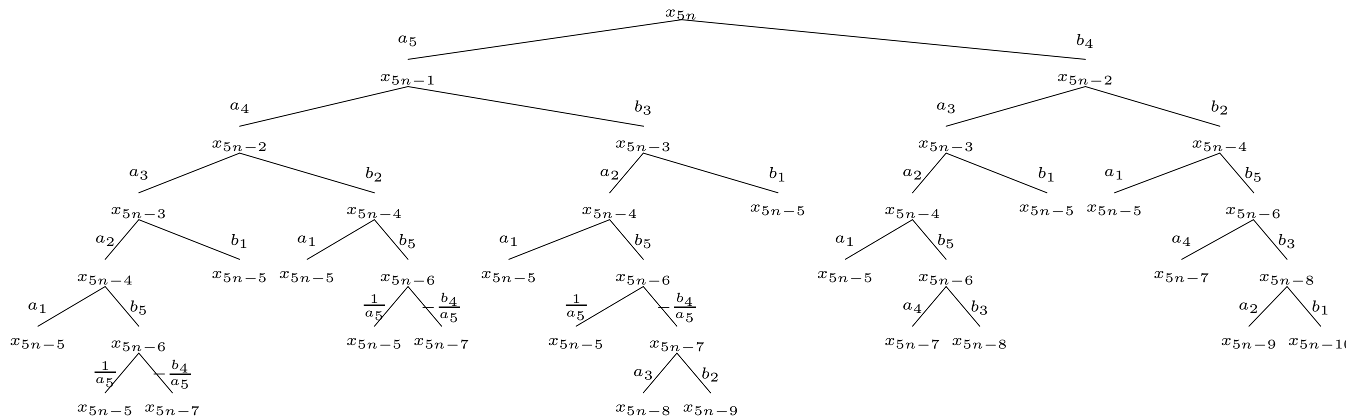
x_{5n}
a_5 b_4
x_{5n-1} x_{5n-2}
a_4 b_3 a_3 b_2
x_{5n-2} x_{5n-3} x_{5n-3} x_{5n-4}
a_3 b_2 a_2 b_1 a_2 b_1 a_1 b_5
x_{5n-3} x_{5n-4} x_{5n-4} x_{5n-5} x_{5n-4} x_{5n-5} x_{5n-5} x_{5n-6}
a_2 b_1 a_1 b_5 a_1 b_5 a_1 b_5 a_4 b_3
x_{5n-4} x_{5n-5} x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-6} x_{5n-7} x_{5n-8}
a_1 b_5 $\frac{1}{a_5}$ $-\frac{b_4}{a_5}$ $\frac{1}{a_5}$ $-\frac{b_4}{a_5}$ a_4 b_3 a_2 b_1
x_{5n-5} x_{5n-6} x_{5n-5} x_{5n-7} x_{5n-5} x_{5n-7} x_{5n-7} x_{5n-8} x_{5n-9} x_{5n-10}
$\frac{1}{a_5}$ $-\frac{b_4}{a_5}$ a_3 b_2
x_{5n-5} x_{5n-7} x_{5n-8} x_{5n-9}

Figure 7.

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AMS Classification Numbers: 11B39, 11B37

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