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IDENTITIES FOR LIKE-POWERS OF LUCAS SEQUENCES FROM ALGEBRAIC IDENTITIES

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Abstract. Let a and b be integers with $b(a^2+4b)\neq 0$. Let $u_0=0$, $u_1=1$, and $u_n=au_{n-1}+bu_{n-2}$ for $n\geq 2$. Let $v_0=2$, $v_1=a$, and $v_n=av_{n-1}+bv_{n-2}$ for $n\geq 2$. Using algebraic identities we will prove some results, including the following ones. For integers $n\geq 0$ and $k\geq 1$,

$$\begin{aligned}
u_{n+3k}^2 &= (v_{2k}+(-b)^k)u_{n+2k}^2-(-b)^k(v_{2k}+(-b)^k)u_{n+k}^2+(-b)^{3k}u_n^2\\
v_{n+3k}^2 &= (v_{2k}+(-b)^k)v_{n+2k}^2-(-b)^k(v_{2k}+(-b)^k)v_{n+k}^2+(-b)^{3k}v_n^2\\
u_{n+4k}^3 &= (v_{3k}+(-b)^k v_k)u_{n+3k}^3-(-b)^k(v_{4k}+(-b)^k v_{2k}+2(-b)^{2k})u_{n+2k}^3\\
&\quad +(-b)^{3k}(v_{3k}+(-b)^k v_k)u_{n+k}^3-(-b)^{6k}u_n^3\\
v_{n+4k}^3 &= (v_{3k}+(-b)^k v_k)v_{n+3k}^3-(-b)^k(v_{4k}+(-b)^k v_{2k}+2(-b)^{2k})v_{n+2k}^3\\
&\quad +(-b)^{3k}(v_{3k}+(-b)^k v_k)v_{n+k}^3-(-b)^{6k}v_n^3.
\end{aligned}$$

These results generalize some results of Gould (1963), Zeitlin and Parker (1963), Bicknell (1972), and Prodinger (1997).

1. Motivation

Definition 1.1. Let a and b be integers with $b(a^2+4b)\neq 0$. Let $u_0=0$, $u_1=1$, and for $n\geq 2$,

$$u_n=au_{n-1}+bu_{n-2}.$$

Let $v_0=2$, $v_1=a$, and for $n\geq 2$,

$$v_n=av_{n-1}+bv_{n-2}.$$

These sequences were originally studied by Lucas [L].

The Binet formulas state that for all $n\geq 0$,

$$u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}\quad\text{and}\quad v_n=\alpha^n+\beta^n,$$

where

$$\alpha=\frac{a+\sqrt{a^2+4b}}{2}\quad\text{and}\quad\beta=\frac{a-\sqrt{a^2+4b}}{2}.$$

These roots are distinct and $\alpha+\beta=a$ and $\alpha\cdot\beta=-b$.

2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37.
Key words and phrases: Lucas sequences, identities, like-powers.
Received 7 August 2016; revised 21 September 2016.
Published online *.

From the Binet formulas, it follows that

$$u^{\ell}_{n+jk} = \sum_{i=0}^{\ell} C_{i,j,k,\ell}\alpha^{in}\beta^{(\ell-i)n},$$

where $C_{i,j,k,\ell}$ are some rational functions in α and β. Fixing ℓ, and giving j values $0, 1, \ldots, \ell+1$, we get $\ell+2$ left-hand sides $u^{\ell}_n, u^{\ell}_{n+k}, \ldots, u^{\ell}_{n+(\ell+1)k}$ such that all of them are linear combinations of the $\ell+1$ monomials from the set $\{\alpha^{in}\beta^{(\ell-i)n};\ 0 \le i \le \ell\}$. By linear algebra, we must have a linear relation among u^{ℓ}_{n+jk} for $0 \le j \le \ell+1$ whose coefficients depend on k and ℓ but not n. Using linear algebra, this linear relation is in fact obtained by expanding the determinant in the right-hand side of the relation

$$0 = \det \begin{pmatrix} C_{0,0,k,\ell} & C_{1,0,k,\ell} & \cdots & C_{\ell,0,k,\ell} & u^{\ell}_n \\ C_{0,1,k,\ell} & C_{1,1,k,\ell} & \cdots & C_{\ell,1,k,\ell} & u^{\ell}_{n+k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{0,\ell+1,k,\ell} & C_{1,\ell+1,k,\ell} & \cdots & C_{\ell,\ell+1,k,\ell} & u^{\ell}_{n+(\ell+1)k} \end{pmatrix}$$

(whose jth row is $(C_{0,j,k,\ell}, C_{1,j,k,\ell}, \ldots, C_{\ell,j,k,\ell}, u^{\ell}_{n+jk})$ for $j = 0, \ldots, \ell+1$) using the last column. Having understood that, it is clear that one gets relations of the sort

$$u^{\ell}_{n+(\ell+1)k} = \sum_{j=0}^{\ell} D_{j,k,\ell} u^{\ell}_{n+jk} \tag{1.1}$$

valid for all n with some coefficients $D_{j,k,\ell}$.

The purpose of this paper is to effectively compute the coefficients of (1.1) when $\ell \le 7$ (and in the analog of (1.1) with the u's replaced by v's).

2. History of identities. Let F_n be the nth Fibonacci number. That is, $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$. Gould [G], Zeitlin and Parker [Z], and Prodinger [P] showed that for any nonnegative integer n,

$$F^2_{n+3} = 2F^2_{n+2} + 2F^2_{n+1} - F^2_n,$$
$$F^3_{n+4} = 3F^3_{n+3} + 6F^3_{n+2} - 3F^3_{n+1} - F^3_n,$$
$$F^4_{n+5} = 5F^4_{n+4} + 15F^4_{n+3} - 15F^4_{n+2} - 5F^4_{n+1} + F^4_n,$$
$$F^5_{n+6} = 8F^5_{n+5} + 40F^5_{n+4} - 60F^5_{n+3} - 40F^5_{n+2} + 8F^5_{n+1} + F^5_n.$$

Bicknell [B] gave a generalization of these results. For this generalization, let p and q be integers and $H_1 = p$, $H_2 = q$, and for any integer $n \ge 1$, let

$$H_{n+2} = H_{n+1} + H_n.$$

Then it was shown that

$$H^2_{n+3} = 2H^2_{n+2} + 2H^2_{n+1} - H^2_n, \qquad H^2_{n+6} = 8H^2_{n+4} - 8H^2_{n+2} + H^2_n.$$

Jarden [J] generalized these results using the u's and v's from Section 1. He showed that for any positive integer k,

$$\sum_{i=0}^{k+1}(-1)^{i(i+1)/2}b^{(i-1)i/2}\binom{k+1}{i}_u u_{n-i}^k = 0,$$

where

$$\binom{m}{j}_u = \begin{cases} 1 & \text{if } j = 0, \\ \dfrac{u_m \cdots u_{m-j+1}}{u_j \cdots u_1} & \text{if } j = 1, \ldots, m. \end{cases}$$

Cooper and Kennedy [C] proved Jarden's result using matrices and eigenvalues.

We will generalize the above results in a different direction. One of our results is that for integers $n \geq 0$ and $k \geq 1$,

$$u_{n+3k}^2 = (v_{2k} + (-b)^k)u_{n+2k}^2 - (-b)^k(v_{2k} + (-b)^k)u_{n+k}^2 + (-b)^{3k}u_n^2,$$
$$v_{n+3k}^2 = (v_{2k} + (-b)^k)v_{n+2k}^2 - (-b)^k(v_{2k} + (-b)^k)v_{n+k}^2 + (-b)^{3k}v_n^2.$$

Our proofs will be algebraic in nature and will involve the use of the Binet formulas for u_n and v_n.

3. Values of $\{u_n\}$ and $\{v_n\}$. The following two tables give the first several values of the sequences $\{u_n\}$ and $\{v_n\}$.

u_n Sequence

	1	b	b^2	b^3	b^4	b^5	b^6	b^7	b^8	b^9
1	1	1	1	1	1	1	1	1	1	1
a	1	2	3	4	5	6	7	8	9	
a^2	1	3	6	10	15	21	28	36	45	
a^3	1	4	10	20	35	56	84	120		
a^4	1	5	15	35	70	126	210	330		
a^5	1	6	21	56	126	252	462			
a^6	1	7	28	84	210	462	924			
a^7	1	8	36	120	330	792				
a^8	1	9	45	165	495	1287				
a^9	1	10	55	220	715					
a^{10}	1	11	66	286	1001					
a^{11}	1	12	78	364						
a^{12}	1	13	91	455						
a^{13}	1	14	105							
a^{14}	1	15	120							
a^{15}	1	16								
a^{16}	1									

The array is Pascal's triangle and the u_n are formed in a manner similar to how the Fibonacci numbers are obtained from Pascal's triangle [O, A000045]. The diagonals of slope 2 give the values of the $\{u_n\}$ sequence by summing the values of the coefficient in the cell times the row label times the column label. The sequence starts at u_1. For example,

$$\begin{aligned}
u_1 &= 1,\\
u_2 &= a,\\
u_3 &= a^2 + b,\\
u_4 &= a^3 + 2ab,\\
u_5 &= a^4 + 3a^2b + b^2,\\
u_6 &= a^5 + 4a^3b + 3ab^2,\\
u_7 &= a^6 + 5a^4b + 6a^2b^2 + b^3.
\end{aligned}$$

v_n Sequence

	1	b	b^2	b^3	b^4	b^5	b^6	b^7	b^8	b^9
1	2	2	2	2	2	2	2	2	2	2
a	1	3	5	7	9	11	13	15	17	
a^2	1	4	9	16	25	36	49	64	81	
a^3	1	5	14	30	55	91	140	204		
a^4	1	6	20	50	105	196	336	540		
a^5	1	7	27	77	182	378	714			
a^6	1	8	35	112	294	672	1386			
a^7	1	9	44	156	450	1122				
a^8	1	10	54	210	660	1782				
a^9	1	11	65	275	935					
a^{10}	1	12	77	352	1287					
a^{11}	1	13	90	442						
a^{12}	1	14	104	546						
a^{13}	1	15	119							
a^{14}	1	16	125							
a^{15}	1	17								
a^{16}	1									

The array is like Pascal's triangle except the first row is all 2's. The v_n are formed in a manner similar to how the Fibonacci numbers are obtained from Pascal's triangle [O, A000045]. The diagonals of slope 2 give the values of the $\{v_n\}$ sequence by summing the values of the coefficient in the cell times the row label times the column label. The sequence starts at v_0. For example,

$$\begin{aligned}
v_0 &= 2,\\
v_1 &= a,\\
v_2 &= a^2 + 2b,\\
v_3 &= a^3 + 3ab,\\
v_4 &= a^4 + 4a^2b + 2b^2,\\
v_5 &= a^5 + 5a^3b + 5ab^2,\\
v_6 &= a^6 + 6a^4b + 9a^2b^2 + 2b^3,\\
v_7 &= a^7 + 7a^5b + 14a^3b^2 + 7ab^3.
\end{aligned}$$

The following theorem and proof will exemplify our results and proofs.

THEOREM 3.1. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be defined as in Definition* 1.1. *Then*

$$\begin{aligned}
u_{n+2k} &= v_k u_{n+k} - (-b)^k u_n,\\
v_{n+2k} &= v_k v_{n+k} - (-b)^k v_n.
\end{aligned}$$

Proof. Using elementary algebra, we can prove that

$$\alpha^{n+2k} \pm \beta^{n+2k} = (\alpha^k + \beta^k)(\alpha^{n+k} \pm \beta^{n+k}) - (\alpha \cdot \beta)^k (\alpha^n \pm \beta^n).$$

Using the "+" signs and Binet's formulas proves the second equation. Using the "−" signs, dividing by $\alpha - \beta$, and using Binet's formula proves the first equation. ■

4. Squares of Lucas sequences. We are now ready to state and prove a result for squares of Lucas sequences.

THEOREM 4.1. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be as defined in Definition* 1.1. *Then*

$$\begin{aligned}
u_{n+3k}^2 &= (v_{2k} + (-b)^k)u_{n+2k}^2 - (-b)^k(v_{2k} + (-b)^k)u_{n+k}^2 + (-b)^{3k}u_n^2,\\
v_{n+3k}^2 &= (v_{2k} + (-b)^k)v_{n+2k}^2 - (-b)^k(v_{2k} + (-b)^k)v_{n+k}^2 + (-b)^{3k}v_n^2.
\end{aligned} \tag{4.1}$$

Proof. Using elementary algebra, we can prove that

$$\begin{aligned}
(\alpha^{n+3k} \pm \beta^{n+3k})^2 = {} & (\alpha^{2k} + \beta^{2k} + (\alpha \cdot \beta)^k)(\alpha^{n+2k} \pm \beta^{n+2k})^2\\
& - (\alpha \cdot \beta)^k(\alpha^{2k} + \beta^{2k} + (\alpha \cdot \beta)^k)(\alpha^{n+k} \pm \beta^{n+k})^2\\
& + (\alpha \cdot \beta)^{3k}(\alpha^n \pm \beta^n)^2.
\end{aligned}$$

Using the "+" signs and Binet's formulas proves the second equation. Using the "–" signs, dividing by $(\alpha-\beta)^2$, and using Binet's formula proves the first equation. ■

5. Special square sequences. We study some of the sequences in equation (4.1) for particular sequences $\{u_n\}$.

In the first example, let n be a nonnegative integer and $u_n = F_n$, where F_n denotes the nth Fibonacci number. For the Fibonacci sequence, $a = 1$ and $b = 1$. Then equation (4.1) becomes

$$F_{n+3k}^2 = (L_{2k} + (-1)^k)F_{n+2k}^2 - (-1)^k(L_{2k} + (-1)^k)F_{n+k}^2 + (-1)^{3k}F_n^2.$$

Here L_n denotes the nth Lucas number. The sequence $\{L_{2n} + (-1)^n\}$ is [O, A047946]. The first few terms of the sequence are:

3, 2, 8, 17, 48, 122, 323, 842, 2208, 5777, 15128, 39602, 103683, 271442, 710648, 1860497, 4870848, 12752042, 33385283, 87403802, 228826128, 599074577, 1568397608, 4106118242, 10749957123, 28143753122, ...

In the second example, let n be a nonnegative integer and $u_n = P_n$, where P_n denotes the nth Pell number. For the Pell sequence, $a = 2$ and $b = 1$. Then equation (4.1) becomes

$$P_{n+3k}^2 = (Q_{2k} + (-1)^k)P_{n+2k}^2 - (-1)^k(Q_{2k} + (-1)^k)P_{n+k}^2 + (-1)^{3k}P_n^2.$$

Here Q_n denotes the nth Pell–Lucas number. The first few terms of the sequence $\{Q_{2n} + (-1)^n\}$ are:

3, 5, 35, 197, 1155, 6725, 39203, 228485, 1331715, 7761797, 45239075, 263672645, 1536796803, 8957108165, 52205852195, ...

For a third example, let n be a nonnegative integer and $u_n = M_n$, where M_n denotes the nth Mersenne number. For the Mersenne sequence, $a = 3$ and $b = -2$. Then equation (4.1) becomes

$$M_{n+3k}^2 = (N_{2k} + 2^k)M_{n+2k}^2 - 2^k(N_{2k} + 2^k)M_{n+k}^2 + 2^{3k}M_n^2.$$

Here N_n denotes the sequence $N_0 = 2$, $N_1 = 3$, and for $n \geq 2$,

$$N_n = 3N_{n-1} - 2N_{n-2}.$$

The sequence $\{N_{2n}+2^n\}$ is [O, A001576]. The first few terms of the sequence are:

3, 7, 21, 73, 273, 1057, 4161, 16513, 65793, 262657, 1049601, 4196353, 16781313, 67117057, 268451841, 1073774593, 4295032833, 17180000257, 68719738881, 274878431233, 1099512676353, ...

6. Cubes of Lucas sequences

THEOREM 6.1. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be defined in Definition* 1.1. *Then*

(6.1)
$$\begin{aligned} u_{n+4k}^3 &= \big(v_{3k} + (-b)^k v_k\big) u_{n+3k}^3 - (-b)^k \big(v_{4k} + (-b)^k v_{2k} + 2(-b)^{2k}\big) u_{n+2k}^3 \\ &\quad + (-b)^{3k} \big(v_{3k} + (-b)^k v_k\big) u_{n+k}^3 - (-b)^{6k} u_n^3 \end{aligned}$$

and

$$\begin{aligned} v_{n+4k}^3 &= \big(v_{3k} + (-b)^k v_k\big) v_{n+3k}^3 - (-b)^k (v_{4k} + (-b)^k v_{2k} + 2(-b)^{2k}) v_{n+2k}^3 \\ &\quad + (-b)^{3k} \big(v_{3k} + (-b)^k v_k\big) v_{n+k}^3 - (-b)^{6k} v_n^3. \end{aligned}$$

Proof. Using elementary algebra, we can prove that

$$\begin{aligned} &(\alpha^{n+4k} \pm \beta^{n+4k})^3 \\ &= \big(\alpha^{3k} + \beta^{3k} + (\alpha \cdot \beta)^k (\alpha^k + \beta^k)\big) (\alpha^{n+3k} \pm \beta^{n+3k})^3 \\ &\quad - (\alpha \cdot \beta)^k \big(\alpha^{4k} + \beta^{4k} + (\alpha \cdot \beta)^k (\alpha^{2k} + \beta^{2k}) + 2(\alpha \cdot \beta)^{2k}\big) (\alpha^{n+2k} \pm \beta^{n+2k})^3 \\ &\quad + (\alpha \cdot \beta)^{3k} \big(\alpha^{3k} + \beta^{3k} + (\alpha \cdot \beta)^k (\alpha^k + \beta^k)\big) (\alpha^{n+k} \pm \beta^{n+k})^3 \\ &\quad - (\alpha \cdot \beta)^{6k} (\alpha^n \pm \beta^n)^3. \end{aligned}$$

Using the "+" signs and Binet's formulas proves the second equation. Using the "−" signs, dividing by $(\alpha - \beta)^3$, and using Binet's formula proves the first equation. ∎

7. Fourth, fifth, and sixth powers of Lucas sequences. We next state identities for the fourth, fifth, and sixth powers of Lucas sequences. The proofs of these theorems follow directly from the corresponding algebraic identity.

THEOREM 7.1. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be defined in Definition* 1.1. *Then*

$$\begin{aligned} u_{n+5k}^4 &= (v_{4k} + (-b)^k v_{2k} + (-b)^{2k}) u_{n+4k}^4 \\ &\quad - (-b)^k \big(v_{6k} + (-b)^k v_{4k} + 2(-b)^{2k} v_{2k} + 2(-b)^{3k}\big) u_{n+3k}^4 \\ &\quad + (-b)^{3k} \big(v_{6k} + (-b)^k v_{4k} + 2(-b)^{2k} v_{2k} + 2(-b)^{3k}\big) u_{n+2k}^4 \\ &\quad - (-b)^{6k} \big(v_{4k} + (-b)^k v_{2k} + (-b)^{2k}\big) u_{n+k}^4 \\ &\quad + (-b)^{10k} u_n^4 \end{aligned}$$

and

$$\begin{aligned}
v_{n+5k}^4 &= \big(v_{4k} + (-b)^k v_{2k} + (-b)^{2k}\big) v_{n+4k}^4 \\
&\quad - (-b)^k \big(v_{6k} + (-b)^k v_{4k} + 2(-b)^{2k} v_{2k} + 2(-b)^{3k}\big) v_{n+3k}^4 \\
&\quad + (-b)^{3k} \big(v_{6k} + (-b)^k v_{4k} + 2(-b)^{2k} v_{2k} + 2(-b)^{3k}\big) v_{n+2k}^4 \\
&\quad - (-b)^{6k} \big(v_{4k} + (-b)^k v_{2k} + (-b)^{2k}\big) v_{n+k}^4 \\
&\quad + (-b)^{10k} v_n^4.
\end{aligned}$$

THEOREM 7.2. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be defined in Definition* 1.1. *Then*

$$\begin{aligned}
u_{n+6k}^5 &= \big(v_{5k} + (-b)^k v_{3k} + (-b)^{2k} v_k\big) u_{n+5k}^5 \\
&\quad - (-b)^k \big(v_{8k} + (-b)^k v_{6k} + 2(-b)^{2k} v_{4k} + 2(-b)^{3k} v_{2k} + 3(-b)^{4k}\big) u_{n+4k}^5 \\
&\quad + (-b)^{3k} \big(v_{9k} + (-b)^k v_{7k} + 2(-b)^{2k} v_{5k} + 3(-b)^{3k} v_{3k} + 3(-b)^{4k} v_k\big) u_{n+3k}^5 \\
&\quad - (-b)^{6k} \big(v_{8k} + (-b)^k v_{6k} + 2(-b)^{2k} v_{4k} + 2(-b)^{3k} v_{2k} + 3(-b)^{4k}\big) u_{n+2k}^5 \\
&\quad + (-b)^{10k} \big(v_{5k} + (-b)^k v_{3k} + (-b)^{2k} v_k\big) u_{n+k}^5 - (-b)^{15k} u_n^5
\end{aligned}$$

and

$$\begin{aligned}
v_{n+6k}^5 &= \big(v_{5k} + (-b)^k v_{3k} + (-b)^{2k} v_k\big) v_{n+5k}^5 \\
&\quad - (-b)^k \big(v_{8k} + (-b)^k v_{6k} + 2(-b)^{2k} v_{4k} + 2(-b)^{3k} v_{2k} + 3(-b)^{4k}\big) v_{n+4k}^5 \\
&\quad + (-b)^{3k} \big(v_{9k} + (-b)^k v_{7k} + 2(-b)^{2k} v_{5k} + 3(-b)^{3k} v_{3k} + 3(-b)^{4k} v_k\big) v_{n+3k}^5 \\
&\quad - (-b)^{6k} \big(v_{8k} + (-b)^k v_{6k} + 2(-b)^{2k} v_{4k} + 2(-b)^{3k} v_{2k} + 3(-b)^{4k}\big) v_{n+2k}^5 \\
&\quad + (-b)^{10k} \big(v_{5k} + (-b)^k v_{3k} + (-b)^{2k} v_k\big) v_{n+k}^5 - (-b)^{15k} v_n^5.
\end{aligned}$$

THEOREM 7.3. *Let $k \geq 1$ and $n \geq 0$ be integers. Let $\{u_n\}$ and $\{v_n\}$ be defined in Definition* 1.1. *Then*

$$\begin{aligned}
u_{n+7k}^6 &= \big(v_{6k} + (b)^k v_{4k} + (-b)^{2k} v_{2k} + (-b)^{3k}\big) u_{n+6k}^6 \\
&\quad - (-b)^k \big(v_{10k} + (-b)^k v_{8k} + 2(-b)^{2k} v_{6k} + 2(-b)^{3k} v_{4k} + 3(-b)^{4k} v_{2k} \\
&\qquad + 3(-b)^{5k}\big) u_{n+5k}^6 \\
&\quad + (-b)^{3k} \big(v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 3(-b)^{3k} v_{6k} + 4(-b)^{4k} v_{4k} \\
&\qquad + 4(-b)^{5k} v_{2k} + 5(-b)^{6k}\big) u_{n+4k}^6 \\
&\quad - (-b)^{6k} \big(v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 3(-b)^{3k} v_{6k} + 4(-b)^{4k} v_{4k} \\
&\qquad + 4(-b)^{5k} v_{2k} + 5(-b)^{6k}\big) u_{n+3k}^6 \\
&\quad + (-b)^{10k} \big(v_{10k} + (-b)^k v_{8k} + 2(-b)^{2k} v_{6k} + 2(-b)^{3k} v_{4k} + 3(-b)^{4k} v_{2k} \\
&\qquad + 3(-b)^{5k}\big) u_{n+2k}^6 \\
&\quad - (-b)^{15k} \big(v_{6k} + (-b)^k v_{4k} + (-b)^{2k} v_{2k} + (-b)^{3k}\big) u_{n+k}^6 \\
&\quad + (-b)^{21k} u_n^6
\end{aligned}$$

and

$$\begin{aligned}
v_{n+7k}^6 = {} & (v_{6k} + (-b)^k v_{4k} + (-b)^{2k} v_{2k} + (-b)^{3k}) v_{n+6k}^6 \\
& - (-b)^k (v_{10k} + (-b)^k v_{8k} + 2(-b)^{2k} v_{6k} + 2(-b)^{3k} v_{4k} + 3(-b)^{4k} v_{2k} \\
& \qquad + 3(-b)^{5k}) v_{n+5k}^6 \\
& + (-b)^{3k} (v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 3(-b)^{3k} v_{6k} + 4(-b)^{4k} v_{4k} \\
& \qquad + 4(-b)^{5k} v_{2k} + 5(-b)^{6k}) v_{n+4k}^6 \\
& - (-b)^{6k} (v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 3(-b)^{3k} v_{6k} + 4(-b)^{4k} v_{4k} \\
& \qquad + 4(-b)^{5k} v_{2k} + 5(-b)^{6k}) v_{n+3k}^6 \\
& + (-b)^{10k} (v_{10k} + (-b)^k v_{8k} + 2(-b)^{2k} v_{6k} + 2(-b)^{3k} v_{4k} + 3(-b)^{4k} v_{2k} \\
& \qquad + 3(-b)^{5k}) v_{n+2k}^6 \\
& - (-b)^{15k} (v_{6k} + (-b)^k v_{4k} + (-b)^{2k} v_{2k} + (-b)^{3k}) v_{n+k}^6 \\
& + (-b)^{21k} v_n^6.
\end{aligned}$$

8. Constructing a seventh powers of Lucas sequence identity. The fourth, fifth, and sixth powers of Lucas sequence identities were discovered by some simple guesses for the corresponding algebraic identity. However, for higher powers, a systematic way forward is needed. We were able to find a seventh powers of Lucas sequence identity using the following method.

In order to find an identity involving seventh powers of Lucas sequences, we will determine constants a's, b's, c's, and d's such that if

$$\begin{aligned}
f = {} & a_1(\alpha^{7k} + \beta^{7k}) + a_2(\alpha \cdot \beta)^k (\alpha^{5k} + \beta^{5k}) + a_3(\alpha \cdot \beta)^{2k} (\alpha^{3k} + \beta^{3k}) \\
& + a_4(\alpha \cdot \beta)^{3k} (a^k + b^k), \\
g = {} & b_1(\alpha^{12k} + \beta^{12k}) + b_2(\alpha \cdot \beta)^k (\alpha^{10k} + \beta^{10k}) + b_3(\alpha \cdot \beta)^{2k} (\alpha^{8k} + \beta^{8k}) \\
& + b_4(\alpha \cdot \beta)^{3k} (\alpha^{6k} + \beta^{6k}) + b_5(\alpha \cdot \beta)^{4k} (\alpha^{4k} + \beta^{4k}) + b_6(\alpha \cdot \beta)^{5k} (\alpha^{2k} + \beta^{2k}) \\
& + b_7(\alpha \cdot \beta)^{6k}, \\
h = {} & c_1(\alpha^{15k} + \beta^{15k}) + c_2(\alpha \cdot \beta)^k (\alpha^{13k} + \beta^{13k}) + c_3(\alpha \cdot \beta)^{2k} (\alpha^{11k} + \beta^{11k}) \\
& + c_4(\alpha \cdot \beta)^{3k} (\alpha^{9k} + \beta^{9k}) + c_5(\alpha \cdot \beta)^{4k} (\alpha^{7k} + \beta^{7k}) + c_6(\alpha \cdot \beta)^{5k} (\alpha^{5k} + \beta^{5k}) \\
& + c_7(\alpha \cdot \beta)^{6k} (\alpha^{3k} + \beta^{3k}) + c_8(\alpha \cdot \beta)^{7k} (\alpha^k + \beta^k), \\
i = {} & d_1(\alpha^{16k} + \beta^{16k}) + d_2(\alpha \cdot \beta)^k (\alpha^{14k} + \beta^{14k}) + d_3(\alpha \cdot \beta)^{2k} (\alpha^{12k} + \beta^{12k}) \\
& + d_4(\alpha \cdot \beta)^{3k} (\alpha^{10k} + \beta^{10k}) + d_5(\alpha \cdot \beta)^{4k} (\alpha^{8k} + \beta^{8k}) \\
& + d_6(\alpha \cdot \beta)^{5k} (\alpha^{6k} + \beta^{6k}) + d_7(\alpha \cdot \beta)^{6k} (\alpha^{4k} + \beta^{4k}) \\
& + d_8(\alpha \cdot \beta)^{7k} (\alpha^{2k} + \beta^{2k}) + d_9(\alpha \cdot \beta)^{8k},
\end{aligned}$$

then

$$(8.1)\quad \begin{aligned}&(\alpha^{n+8k}+\beta^{n+8k})^7\\ &= f\cdot(\alpha^{n+7k}+\beta^{n+7k})^7-(\alpha\cdot\beta)^k\cdot g\cdot(\alpha^{n+6k}+\beta^{n+6k})^7\\ &\quad+(\alpha\cdot\beta)^{3k}\cdot h\cdot(\alpha^{n+5k}+\beta^{n+5k})^7-(\alpha\cdot\beta)^{6k}\cdot i\cdot(\alpha^{n+4k}+\beta^{n+4k})^7\\ &\quad+(\alpha\cdot\beta)^{10k}\cdot h\cdot(\alpha^{n+3k}+\beta^{n+3k})^7-(\alpha\cdot\beta)^{15k}\cdot g\cdot(\alpha^{n+2k}+\beta^{n+2k})^7\\ &\quad+(\alpha\cdot\beta)^{21k}\cdot f\cdot(\alpha^{n+k}+\beta^{n+k})^7-(\alpha\cdot\beta)^{28k}\cdot(\alpha^n+\beta^n)^7.\end{aligned}$$

Expanding the LHS of (8.1) we have

$$\begin{aligned}&a^{7n+56k}+7a^{6n+48k}b^{n+8k}+21a^{5n+40k}b^{2n+16k}+35a^{4n+32k}b^{3n+24k}\\ &\quad+35a^{3n+24k}b^{4n+32k}+21a^{2n+16k}b^{5n+40k}+7a^{n+8k}b^{6n+48k}+b^{7n+56k}.\end{aligned}$$

Expanding the RHS of (8.1) is quite a lengthy task. However, this must be done to determine the a's, b's, c's, and d's. Our first task is to determine the coefficients of all the terms of the form a^{7n+xk} on the RHS of the equation where $x = 56, 55, \ldots, 42$. The other x values give redundant equations. Equating the coefficients of the LHS and RHS of the equation results in a set of equations involving the a's, b's, c's, and d's. This set of equations is

1. $a_1 = 1,$
2. $a_2 - b_1 = 0,$
3. $a_3 - b_2 = 0,$
4. $a_4 - b_3 + c_1 = 0,$
5. $a_4 - b_4 + c_2 = 0,$
6. $a_3 - b_5 + c_3 = 0,$
7. $a_2 - b_6 + c_4 - d_1 = 0,$
8. $a_1 - b_7 + c_5 - d_2 = 0,$
9. $-b_6 + c_6 - d_3 = 0,$
10. $-b_5 + c_7 - d_4 = 0,$
11. $-b_4 + c_8 - d_5 + c_1 = 0,$
12. $-b_3 + c_8 - d_6 + c_2 = 0,$
13. $-b_2 + c_7 - d_7 + c_3 = 0,$
14. $-b_1 + c_6 - d_8 + c_4 = 0,$
15. $c_5 - d_9 + c_5 = 0.$

Next, we determine the coefficients of all the terms of the form $a^{6n+xk}b^{n+yk}$ on the RHS of the equation where $x = 49, 48, \ldots, 38$. The other x values give redundant equations. Equating the coefficients of the LHS and RHS of the equation results in a second set of equations involving the a's, b's, c's, and d's. This set of equations is

16. $a_1 - b_1 = 0,$
17. $a_2 - b_2 + c_1 = 1,$
18. $a_3 - b_3 + c_2 = 0,$
19. $a_4 - b_4 + c_3 - d_1 = 0,$
20. $a_4 - b_5 + c_4 - d_2 = 0,$
21. $a_3 - b_6 + c_5 - d_3 = 0,$
22. $a_2 - b_7 + c_6 - d_4 + c_1 = 0,$
23. $a_1 - b_6 + c_7 - d_5 + c_2 = 0,$
24. $-b_5 + c_8 - d_6 + c_3 = 0,$
25. $-b_4 + c_8 - d_7 + c_4 = 0,$
26. $-b_3 + c_7 - d_8 + c_5 - b_1 = 0,$
27. $-b_2 + c_6 - d_9 + c_6 - b_2 = 0.$

Next, we determine the coefficients of all the terms of the form $a^{5n+xk}b^{2n+yk}$ on the RHS of the equation where $x = 43, 42, \ldots, 33$. The other x values give redundant equations. Equating the coefficients of the LHS and RHS of the equation results in a third set of equations involving the a's, b's, c's, and d's. This set of equations is

28. $-b_1 + c_1 = 0,$
29. $a_1 - b_2 + c_2 - d_1 = 0,$
30. $a_2 - b_3 + c_3 - d_2 = 0,$
31. $a_3 - b_4 + c_4 - d_3 + c_1 = 0,$
32. $a_4 - b_5 + c_5 - d_4 + c_2 = 0,$
33. $a_4 - b_6 + c_6 - d_5 + c_3 = 0,$
34. $a_3 - b_7 + c_7 - d_6 + c_4 - b_1 = 0,$
35. $a_2 - b_6 + c_8 - d_7 + c_5 - b_2 = 0,$
36. $a_1 - b_5 + c_8 - d_8 + c_6 - b_3 = 0,$
37. $-b_4 + c_7 - d_9 + c_7 - b_4 = 0.$

Finally, we determine the coefficients of all the terms of the form $a^{4n+xk}b^{3n+yk}$ on the RHS of the equation where $x = 38, 37, \ldots, 30$. The other x values give redundant equations. Equating the coefficients of the LHS and RHS of the equation results in a fourth set of equations involving the a's, b's, c's, and d's. This set of equations is

38. $c_1 - d_1 = 0,$
39. $-b_1 + c_2 - d_2 + a_1 = 0,$

40. $-b_2 + c_3 - d_3 + a_2 = 0,$
41. $a_1 - b_3 + c_4 - d_4 + c_3 - b_1 = 0,$
42. $a_2 - b_4 + c_5 - d_5 + c_4 - b_2 = 0,$
43. $a_3 - b_5 + c_6 - d_6 + c_5 - b_3 = 0,$
44. $a_4 - b_6 + c_7 - d_7 + c_6 - b_4 + a_1 = 1,$
45. $a_4 - b_7 + c_8 - d_8 + c_7 - b_5 + a_2 = 0,$
46. $a_3 - b_6 + c_8 - d_9 + c_8 - b_6 + a_3 = 0.$

The equations for the coefficients of all the terms of the form $a^{3n+xk}b^{4n+yk}$ are the same as the fourth set of equations. The equations for the coefficients of all the terms of the form $a^{2n+xk}b^{5n+yk}$ are the same as the third set of equations. The equations for the coefficients of all the terms of the form $a^{n+xk}b^{6n+yk}$ are the same as the second set of equations. And the equations for the coefficients of all the terms of the form b^{7n+yk} are the same as the first set of equations.

Many of the above equations listed are dependent. Using Maxima, equations 39, 40, 9, 32, 41, 10, 37, 42, 33, 46, 45, 36, 35, 13, 44, 43, 12, 34 were eliminated as dependent equations. And solving the remaining set of equations yields a unique solution. That solution is

i	1	2	3	4	5	6	7	8	9
a_i	1	1	1	1					
b_i	1	1	2	2	3	3	4		
c_i	1	1	2	3	4	5	6	6	
d_i	1	1	2	3	5	5	7	7	8

Therefore, the identity for the seventh powers of Lucas sequences is

$$\begin{aligned}
v_{n+8k}^7 = {} & (v_{7k} + (-b)^k v_{5k} + (-b)^{2k} v_{3k} + (-b)^{3k} v_k) v_{n+7k}^7 \\
& - (-b)^k (v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 2(-b)^{3k} v_{6k} + 3(-b)^{4k} v_{4k} \\
& \qquad + 3(-b)^{5k} v_{2k} + 4(-b)^{6k}) v_{n+6k}^7 \\
& + (-b)^{3k} (v_{15k} + (-b)^k v_{13k} + 2(-b)^{2k} v_{11k} + 3(-b)^{3k} v_{9k} + 4(-b)^{4k} v_{7k} \\
& \qquad + 5(-b)^{5k} v_{5k} + 6(-b)^{6k} v_{3k} + 6(-b)^{7k} v_k) v_{n+5k}^7 \\
& - (-b)^{6k} (v_{16k} + (-b)^k v_{14k} + 2(-b)^{2k} v_{12k} + 3(-b)^{3k} v_{10k} + 5(-b)^{4k} v_{8k} \\
& \qquad + 5(-b)^{5k} v_{6k} + 7(-b)^{6k} v_{4k} + 7(-b)^{7k} v_{2k} + 8(-b)^{8k}) v_{n+4k}^7 \\
& + (-b)^{10k} (v_{15k} + (-b)^k v_{13k} + 2(-b)^{2k} v_{11k} + 3(-b)^{3k} v_{9k} + 4(-b)^{4k} v_{7k} \\
& \qquad + 5(-b)^{5k} v_{5k} + 6(-b)^{6k} v_{3k} + 6(-b)^{7k} v_k) v_{n+3k}^7
\end{aligned}$$

$$- (-b)^{15k}(v_{12k} + (-b)^k v_{10k} + 2(-b)^{2k} v_{8k} + 2(-b)^{3k} v_{6k} + 3(-b)^{4k} v_{4k} + 3(-b)^{5k} v_{2k} + 4(-b)^{6k}) v_{n+2k}^7$$
$$+ (-b)^{21k}(v_{7k} + (-b)^k v_{5k} + (-b)^{2k} v_{3k} + (-b)^{3k} v_k) v_{n+k}^7$$
$$- (-b)^{28k} v_n^7.$$

A similar identity follows for the $\{u_n\}$.

Acknowledgments. The author wishes to express his thanks to an anonymous referee for suggestions which greatly improved the paper.

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