LUCAS $(a_1, a_2, \ldots, a_k = 1)$ SEQUENCES AND PSEUDOPRIMES

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Abstract. Bisht defined a generalized Lucas integral sequence of order $k \geq 1$ for nonnegative integers n as

$$G_n = x_1^n + x_2^n + \cdots + x_k^n,$$

where x_1, x_2, …, x_k are the roots of the equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k$$

with integral coefficients and $a_k \neq 0$. He proved that these sequences satisfy the congruence

$$G_p \equiv G_1 \pmod{p}$$

when p is prime. Imposing the condition $a_k = 1$, we extend these generalized Lucas integral sequences to negative indices and define these sequences as Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequences. We then prove that

$$G_{-p} \equiv G_{-1} \pmod{p}$$

when p is prime. Finally, we define the concept of Lucas $(a_1, a_2, \ldots, a_k = 1)$ pseudoprime and study some particular examples and prove some theorems.

1. Definitions and Closed Form

Bisht [6] defined a generalized Lucas integral sequence of order $k \geq 1$ as follows:

Definition 1.1. *Let n be a nonnegative integer and*

$$G_n = x_1^n + x_2^n + \cdots + x_k^n,$$

where x_1, x_2, …, x_k are the roots of the equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k$$

with integral coefficients and $a_k \neq 0$.

A generalized Lucas integral sequence of order 1 is just $G_n = a_1^n$ for nonnegative integers n. The generalized Lucas integral sequence of order 2 with equation $x^2 = x + 1$ is just $G_n = L_n$, where L_n is the nth Lucas number. Perrin's sequence (Sloane [21] - A001608) is the generalized Lucas integral sequence of order 3 with $a_1 = 0$, $a_2 = 1$, and $a_3 = 1$. Alternately, Perrin's sequence can be defined as $G_0 = 3$, $G_1 = 0$, $G_2 = 2$, and

$$G_n = G_{n-2} + G_{n-3} \quad \text{for } n \geq 3.$$

Bisht [6], using Newton's formulas [24], proved the following alternate definition of a generalized Lucas integral sequence of order $k \geq 1$.

Definition 1.2. *Let $a_1, \ldots, a_k$ be integers and $a_k \neq 0$. Let*

$$\begin{aligned} G_0 &= k, \\ G_n &= \textstyle\sum_{j=1}^{n-1} a_j G_{n-j} + n a_n, \quad \text{if } n = 1, 2, \ldots, k-1, \\ G_n &= \textstyle\sum_{j=1}^{k} a_j G_{n-j}, \quad \text{if } n \geq k. \end{aligned}$$

The generalized Lucas integral sequences of order $k \geq 1$ have a closed form related to the Girard-Waring formula. A discussion of the Girard-Waring formula can be found in [9].

Theorem 1.1 (Girard-Waring Formula)**.** *Let k be a positive integer and let x_1, x_2, $\ldots$, x_k be the roots of the polynomial*

$$x^k + b_1 x^{k-1} + b_2 x^{k-2} + \cdots + b_k.$$

For $j \geq 0$ an integer, define

$$s_j = \sum_{i=1}^{k} x_i^j.$$

Then for every positive integer n,

$$s_n = n \cdot \sum_{\substack{i_1+2i_2+\cdots+ki_k=n \\ i_1,i_2,\ldots,i_k \geq 0}} (-1)^{i_1+i_2+\cdots+i_k} \frac{(i_1+i_2+\cdots+i_k-1)!}{i_1! i_2! \cdots i_k!} b_1^{i_1} b_2^{i_2} \cdots b_k^{i_k}.$$

Thus, we have the following corollary.

Corollary 1.2 (Closed Form)**.** *Let $\{G_n\}$ be a generalized Lucas integral sequence of order $k \geq 1$ and n be a positive integer. Then*

$$G_n = n \cdot \sum_{\substack{i_1+2i_2+\cdots+ki_k=n \\ i_1,i_2,\ldots,i_k \geq 0}} \frac{(i_1+i_2+\cdots+i_k-1)!}{i_1! i_2! \cdots i_k!} a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k}.$$

Proof. Let $\{G_n\}$ be a generalized Lucas integral sequence of order $k \geq 1$ and assume that x_1, x_2, $\ldots$, x_k are the roots of the equation

$$(x - x_1)(x - x_2) \cdots (x - x_k) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_k.$$

Then applying the Girard-Waring formula and simplifying, we obtain the result. □

2. Divisibility

Bisht [6] proved the following theorem.

Theorem 2.1. *Let $\{G_n\}$ be a generalized Lucas integral sequence of order $k \geq 1$ and p be a prime number. Then for positive integers m and r,*

$$G_{mp^r} \equiv G_{mp^{r-1}} \pmod{p^r}.$$

Historically, we need to mention the following results. For any generalized Lucas integral sequence of order 1, the congruence $G_{p^r} \equiv G_{p^{r-1}} \pmod{p^r}$, which is a special case of Theorem 2.1, is just Fermat's Little Theorem. Hoggatt and Bicknell [13] proved that if p is prime, then $L_p \equiv L_1 \pmod{p}$. Lucas [16] studied Perrin's sequence and proved that if p is prime, then $G_p \equiv G_1 \pmod{p}$.

3. Lucas $(a_1, a_2, \ldots, a_k = 1)$ Sequences

During their study of Perrin's sequence, Adams and Shanks [1] extended Perrin's sequence to negative indices and proved that $G_{-p} \equiv G_{-1} \pmod{p}$ when p is prime. We wish to extend the definition of a generalized Lucas integral sequence of order $k \geq 1$ to negative indices so that

$$G_n = x_1^n + x_2^n + \cdots + x_k^n$$

is true for negative integers n. In the meantime, we want these G_n's to be integers. One way to do this is to let $a_k = 1$. Therefore, we define G for negative indices as follows:

Definition 3.1. *Let $\{G_n\}$ be a generalized Lucas integral sequence of order $k \geq 1$ with $a_k = 1$. Let*

$$\begin{aligned} G_0 &= k, \\ G_{-n} &= -\textstyle\sum_{j=1}^{n-1} a_{k-j} G_{-n+j} - n a_{k-n}, \quad \text{if } n = 1, 2, \ldots, k-1, \\ G_{-n} &= -\textstyle\sum_{j=1}^{k-1} a_{k-j} G_{-n+j} + G_{-n+k}, \quad \text{if } n \geq k. \end{aligned}$$

We note that a similar definition to Definition 3.1 can be given for the Lucas $(a_1, a_2, \ldots, a_k = -1)$ sequence.

At this point, it is helpful to compute several examples of Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequences.

Example 3.1 (Lucas $(1, 1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
G_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521
G_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123	-199	322	-521

This is the Lucas sequence.

Example 3.2 (Lucas $(2, 1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11
G_n	2	2	6	14	34	82	198	478	1154	2786	6726	16238
G_{-n}	2	-2	6	-14	34	-82	198	-478	1154	-2786	6726	-16238

This is the Pell-Lucas sequence.

Example 3.3 (Lucas $(0,1,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
G_n	3	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68	90	119
G_{-n}	3	-1	1	2	-3	4	-2	-1	5	-7	6	-1	-6	12	-13	7	5	-18

This is Perrin's sequence.

Example 3.4 (Lucas $(0,2,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
G_n	3	0	4	3	8	10	19	28	48	75	124	198	323	520	844
G_{-n}	3	-2	4	-5	8	-12	19	-30	48	-77	124	-200	323	-522	844

In this example, $G_{2n+1}(0,2,1) = G_{2n+1}(1,1) - 1$ and $G_{2n}(0,2,1) = G_{2n}(1,1) + 1$ for n an integer.

Example 3.5 (Lucas $(1,0,1,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
G_n	4	1	1	4	9	11	16	29	49	76	121	199	324	521	841
G_{-n}	4	-1	1	-4	9	-11	16	-29	49	-76	121	-199	324	-521	841

In this example, $G_{2n+1}(1,0,1,1) = G_{2n+1}(1,1)$, $G_{4n}(1,0,1,1) = G_{4n}(1,1) + 2$, and $G_{4n+2}(1,0,1,1) = G_{4n+2}(1,1) - 2$ for n an integer.

Example 3.6 (Lucas $(0,2,0,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
G_n	4	0	4	0	12	0	28	0	68	0	164	0	396	0	956	0
G_{-n}	4	0	-4	0	12	0	-28	0	68	0	-164	0	396	0	-956	0

Example 3.7 (Lucas $(0,4,0,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
G_n	4	0	8	0	36	0	152	0	644	0	2728	0	6100	0	14928
G_{-n}	4	0	-8	0	36	0	-152	0	644	0	-2728	0	6100	0	-14928

Example 3.8 (Lucas $(1,3,1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12
G_n	3	1	7	13	35	81	199	477	1155	2785	6727	16237	39203
G_{-n}	3	-3	7	-15	35	-83	199	-479	1155	-2787	6727	-16239	39203

In this example, $G_{2n+1}(1,3,1) = G_{2n+1}(2,1) - 1$ and $G_{2n}(1,3,1) = G_{2n}(2,1) + 1$ for n an integer.

Example 3.9 (Lucas $(2, 0, 2, 1)$ Sequence)**.**

n	0	1	2	3	4	5	6	7	8	9	10	11	12
G_n	4	2	4	14	36	82	196	478	1156	2786	6724	16238	39204
G_{-n}	4	-2	4	-14	36	-82	196	-478	1156	-2786	6724	-16238	39204

In this example, $G_{2n+1}(2,0,2,1) = G_{2n+1}(2,1)$, $G_{4n}(2,0,2,1) = G_{4n}(2,1) + 2$, and $G_{4n+2}(2,0,2,1) = G_{4n+2}(2,1) - 2$ for n an integer.

4. Results

Theorem 4.1. *Let $\{G_n\}$ be a Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequence and let p be a prime. Then for positive integers m and r,*

$$G_{-mp^r} \equiv G_{-mp^{r-1}} \pmod{p^r}.$$

Proof. Let $\{G_n\}$ be a Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequence. Define the generalized Lucas integral sequence of order $k \geq 1$ $\{H_n\}$ as

$$H_n = y_1^n + y_2^n + \cdots + y_k^n,$$

where y_1, y_2, …, y_k are the roots of the equation

$$y^k = -a_{k-1}y^{k-1} - \cdots - a_1 y + 1.$$

But the y_i's are just the reciprocals of the x_i's, the roots of the equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_{k-1}x + 1.$$

Thus, $H_n = G_{-n}$ for all nonnegative integers n. Using the same argument as Bisht [6], for positive integers m and r and for p a prime

$$H_{mp^r} \equiv H_{mp^{r-1}} \pmod{p^r}.$$

Therefore,

$$G_{-mp^r} \equiv G_{-mp^{r-1}} \pmod{p^r}$$

and the proof is complete. □

We note that a similar theorem to Theorem 4.1 can be proved for the Lucas $(a_1, a_2, \ldots, a_k = -1)$ sequence.

We also note that for every Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequence $\{G_n\}$, $G_1 = a_1$ and $G_{-1} = -a_{k-1}$.

Next, we give some more general examples of Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequences.

Example 4.1. *Let $\{G_n\}$ be a Lucas $(0, a_2, \ldots, a_{k-2}, 0, a_k = 1)$ sequence. Then by Theorem 2.1 and Theorem 4.1, the sequence $\{G_n\}$ satisfies the congruences*

$$G_p \equiv 0 \pmod{p} \quad \textit{and} \quad G_{-p} \equiv 0 \pmod{p},$$

where p is prime.

Example 4.2. *Let $\{G_n\}$ be a Lucas $(0, 1, \ldots, 1, a_k = 1)$ sequence. When $k = 3$ this is Perrin's sequence. The initial conditions for these sequences are:*

n	0	1	2	3	4	5	6	$\cdots$	$k-1$
G_n	k	0	2	3	6	10	17	$\cdots$	$L_{k-1}-1$
G_{-n}	k	-1	-1	-1	-1	-1	-1	$\cdots$	$k-2$

Example 4.3. *Let $\{G_n\}$ be a Lucas $(1, 1, \ldots, 1, a_k = 1)$ sequence. This is a Fibonacci-like sequence. When $k = 2$ this sequence is the Lucas sequence. When $k = 3$ this is the sequence (Sloane [21] - A001644). The initial conditions for these sequences are:*

n	0	1	2	3	4	5	6	$\cdots$	$k-1$
G_n	k	1	3	7	15	31	63	$\cdots$	$2^{k-1}-1$
G_{-n}	k	-1	-1	-1	-1	-1	-1	$\cdots$	-1

5. Pseudoprimes

The concept of a pseudoprime with respect to a polynomial has been studied in the mathematical literature by Gurak [12], Szekeres [23], Atkin [4], and Grantham [10]. We note that the Frobenius pseudoprimes of Grantham [10] generalize the higher-order pseudoprimes of both Gurak and Szekeres. The higher-order pseudoprime test of Atkin [4] shows some similarities to the Frobenius pseudoprime test of Grantham. Motivated by the results above, we make the following definition.

Definition 5.1. *Let $\{G_n\}$ be a Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequence. A composite n such that $G_n \equiv G_1 \pmod{n}$ and $G_{-n} \equiv G_{-1} \pmod{n}$ is called a Lucas $(a_1, a_2, \ldots, a_k = 1)$ pseudoprime.*

Again, we note that a similar definition to Definition 5.1 can be given for the Lucas $(a_1, a_2, \ldots, a_k = -1)$ sequence.

Traditional Lucas $(a_1, 1)$ pseudoprimes are essentially Lucas $(a_1, 1)$ pseudoprimes. However, a traditional Lucas $(a_1, 1)$ pseudoprime only requires that n is composite and $G_n \equiv G_1 \pmod{n}$. Therefore, every Lucas $(a_1, 1)$ pseudoprime is a traditional Lucas $(a_1, 1)$ pseudoprime. But these pseudoprimes are different. For example, 8 is a traditional Lucas $(2, 1)$ pseudoprime since $G_8 = 1154$, $G_1 = 2$, and $1154 \equiv 2 \pmod{8}$. However, 8 is not a Lucas $(2, 1)$ pseudoprime with the above definition since $G_{-8} = 1154$, $G_{-1} = -2$, and $1154 \not\equiv -2 \pmod{8}$. A similar argument can be made for any 2^k, where $k \geq 3$. It should be noted that Beeger [5] proved there are an infinite number of even m such that $2^m \equiv 2 \pmod{m}$.

The following theorem gives some results about traditional Lucas $(a_1, 1)$ pseudoprimes and Lucas $(a_1, 1)$ pseudoprimes. The proof of the theorem will appear in a separate paper.

Theorem 5.1. *Every odd traditional Lucas $(a_1, 1)$ pseudoprime is a Lucas $(a_1, 1)$ pseudoprime. If $a_1 \equiv 2 \pmod{4}$, then 4 is a Lucas $(a_1, 1)$ pseudoprime. There are only a finite number of even Lucas $(a_1, 1)$ pseudoprimes.*

We also note that traditional Lucas $(a_1, -1)$ pseudoprimes are Lucas $(a_1, -1)$ pseudoprimes.

In the case of Perrin's sequence (Lucas $(0, 1, 1)$ sequence), Perrin [18] searched for a composite n such that $n|G_n$. Adams and Shanks [1] extensively studied Perrin's sequence and its properties. They found that the smallest composite n such that $n|G_n$ is 271441. Every Lucas $(0, 1, 1)$ pseudoprime is a Perrin pseudoprime. However, 271441 is not a Lucas $(0, 1, 1)$ pseudoprime.

We wrote a program to find examples of Lucas $(a_1, a_2, \ldots, a_k = \pm 1)$ pseudoprimes. Here are some of our findings. The program can be found in the Appendix.

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
2	$(0, 1)$	$9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65 \leq 67$
2	$(1, 1)$	$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201 \leq 15000$
2	$(2, 1)$	$4, 169, 385, 961, 1105, 1121, 3827, 4901, 6265, 6441, 6601, 7107 \leq 7500$
2	$(3, 1)$	$33, 65, 119, 273, 377, 385, 533, 561, 649, 1105, 1189, 1441 \leq 1500$
2	$(4, 1)$	$9, 85, 161, 341, 705, 897, 901, 1105, 1281, 1853, 2465 \leq 2500$
2	$(5, 1)$	$9, 27, 65, 121, 145, 377, 385, 533, 1035, 1189, 1305, 1885 \leq 2000$
2	$(6, 1)$	$4, 57, 185, 385, 481, 629, 721, 779, 1121, 1441, 1729 \leq 2000$
2	$(0, -1)$	$9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65 \leq 67$
2	$(1, -1)$	$25, 35, 49, 55, 65, 77, 85, 91, 95, 115, 119, 121, 125 \leq 130$
2	$(2, -1)$	$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28 \leq 29$
2	$(3, -1)$	$4, 15, 44, 105, 195, 231, 323, 377, 435, 665, 705, 836, 1364 \leq 1400$
2	$(4, -1)$	$10, 209, 230, 231, 399, 430, 455, 530, 901, 903, 923, 9891295 \leq 1500$
2	$(5, -1)$	$15, 21, 35, 105, 161, 195, 255, 345, 385, 399, 465, 527, 551 \leq 600$
2	$(6, -1)$	$4, 14, 28, 35, 119, 164, 169, 385, 434, 574, 741, 779, 899 \leq 900$
3	$(0, 0, 1)$	$4, 8, 10, 14, 16, 20, 22, 25, 26, 28, 32, 34, 35, 38, 40 \leq 42$
3	$(1, 0, 1)$	≤ 1000000
3	$(0, 1, 1)$	≤ 10000000
3	$(2, 0, 1)$	≤ 1000000
3	$(1, 1, 1)$	≤ 1000000
3	$(0, 2, 1)$	$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201 \leq 15000$

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
3	$(3, 0, 1)$	≤ 100000
3	$(2, 1, 1)$	≤ 100000
3	$(1, 2, 1)$	$4 \leq 100000$
3	$(0, 3, 1)$	≤ 100000
3	$(4, 0, 1)$	$4 \leq 100000$
3	$(3, 1, 1)$	$4, 66, 33153, 79003 \leq 100000$
3	$(2, 2, 1)$	$79003 \leq 100000$
3	$(1, 3, 1)$	$169, 385, 961, 1105, 1121, 3827, 4901, 6265, 6441, 6601 \leq 7000$
3	$(0, 4, 1)$	$4 \leq 100000$
3	$(5, 0, 1)$	$470, 2465, 10585, 12801, 15457, 15841 \leq 100000$
3	$(4, 1, 1)$	$75361 \leq 100000$
3	$(3, 2, 1)$	≤ 100000
3	$(2, 3, 1)$	$4, 8 \leq 100000$
3	$(1, 4, 1)$	≤ 100000
3	$(0, 5, 1)$	$946 \leq 100000$
3	$(6, 0, 1)$	≤ 100000
3	$(5, 1, 1)$	$289 \leq 100000$
3	$(4, 2, 1)$	≤ 100000
3	$(3, 3, 1)$	$6 \leq 100000$
3	$(2, 4, 1)$	$33, 65, 119, 273, 377, 385, 533, 561, 649, 1105, 1189, 1441 \leq 2000$
3	$(1, 5, 1)$	≤ 100000
3	$(0, 6, 1)$	≤ 100000
3	$(0, 0, -1)$	$4, 8, 10, 14, 16, 20, 22, 25, 26, 28, 32, 34, 35, 58, 40, 44 \leq 45$
3	$(1, 0, -1)$	≤ 100000
3	$(0, 1, -1)$	≤ 100000
3	$(2, 0, -1)$	$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251 \leq 20000$
3	$(1, 1, -1)$	$9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65, 69 \leq 70$
3	$(0, 2, -1)$	$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251 \leq 20000$
3	$(3, 0, -1)$	≤ 100000
3	$(2, 1, -1)$	$4 \leq 100000$
3	$(1, 2, -1)$	$4 \leq 100000$
3	$(0, 3, -1)$	≤ 100000

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
3	$(4, 0, -1)$	$4 \leq 100000$
3	$(3, 1 - 1)$	≤ 100000
3	$(2, 2, -1)$	$15, 105, 195, 231, 323, 377, 435, 665, 705, 1443, 1551, 1891, 2465 \leq 2500$
3	$(1, 3, -1)$	≤ 100000
3	$(0, 4, -1)$	$4 \leq 100000$
3	$(5, 0, -1)$	≤ 100000
3	$(4, 1, -1)$	$25 \leq 100000$
3	$(3, 2, -1)$	≤ 100000
3	$(2, 3, -1)$	≤ 100000
3	$(1, 4, -1)$	$25 \leq 100000$
3	$(0, 5, -1)$	≤ 100000
3	$(6, 0, -1)$	≤ 100000
3	$(5, 1, -1)$	≤ 100000
3	$(4, 2, -1)$	$9, 27 \leq 100000$
3	$(3, 3, -1)$	$4, 6, 8, 12, 16, 18, 24, 30, 32, 36, 48, 54, 56, 60, 64, 72 \leq 85$
3	$(2, 4, -1)$	$9, 27 \leq 100000$
3	$(1, 5, -1)$	≤ 100000
3	$(0, 6, -1)$	≤ 100000

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
4	$(0, 0, 0, 1)$	$4, 6, 9, 10, 14, 15, 18, 21, 22, 25, 26, 27, 30, 33, 34, 35 \leq 37$
4	$(1, 0, 0, 1)$	$4, 34, 38, 46, 62, 94, 106, 122, 158, 166, 214, 218, 226 \leq 273$
4	$(0, 1, 0, 1)$	$9, 12, 15, 21, 25, 27, 33, 35, 36, 39, 45, 49, 51, 55, 57 \leq 60$
4	$(0, 0, 1, 1)$	≤ 100000
4	$(2, 0, 0, 1)$	≤ 100000
4	$(0, 2, 0, 1)$	$4, 9, 12, 15, 21, 25, 27, 33, 35, 36, 39, 45, 49, 51, 55, 57 \leq 60$
4	$(0, 0, 2, 1)$	$6 \leq 100000$
4	$(1, 1, 0, 1)$	≤ 100000
4	$(1, 0, 1, 1)$	$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201 \leq 15000$
4	$(0, 1, 1, 1)$	≤ 100000
4	$(3, 0, 0, 1)$	≤ 100000
4	$(0, 3, 0, 1)$	$6, 9, 15, 18, 21, 25, 27, 33, 35, 39, 45, 49, 51, 54, 55, 57 \leq 60$
4	$(0, 0, 3, 1)$	$4 \leq 100000$
4	$(2, 1, 0, 1)$	≤ 100000
4	$(2, 0, 1, 1)$	≤ 100000
4	$(1, 2, 0, 1)$	$4 \leq 100000$
4	$(1, 0, 2, 1)$	≤ 100000
4	$(0, 2, 1, 1)$	≤ 100000
4	$(0, 1, 2, 1)$	$2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251 \leq 25000$
4	$(1, 1, 1, 1)$	$49 \leq 100000$
4	$(4, 0, 0, 1)$	$4, 6 \leq 100000$
4	$(0, 4, 0, 1)$	$4, 9, 12, 15, 21, 25, 27, 33, 35, 36, 39, 45, 49, 51, 55, 57 \leq 60$
4	$(0, 0, 4, 1)$	$4, 10, \leq 100000$
4	$(3, 1, 0, 1)$	≤ 100000
4	$(3, 0, 1, 1)$	$9 \leq 100000$
4	$(1, 3, 0, 1)$	≤ 100000
4	$(1, 0, 3, 1)$	$4, 9 \leq 100000$
4	$(0, 3, 1, 1)$	≤ 100000
4	$(0, 1, 3, 1)$	≤ 100000
4	$(2, 2, 0, 1)$	≤ 100000
4	$(2, 0, 2, 1)$	$169, 385, 961, 1105, 1121, 3827, 4901, 6265, 6441, 6601, 7107 \leq 7500$
4	$(0, 2, 2, 1)$	≤ 100000

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
4	$(2, 1, 1, 1)$	$4 \leq 100000$
4	$(1, 2, 1, 1)$	≤ 100000
4	$(1, 1, 2, 1)$	≤ 100000
4	$(0, 0, 0, -1)$	$4, 6, 9, 10, 14, 15, 18, 21, 22, 25, 26, 27, 30, 33, 34, 35 \leq 37$
4	$(1, 0, 0, -1)$	$4, 34, 38, 46, 62, 94, 106, 122, 158, 166, 214, 218, 226, 274 \leq 277$
4	$(0, 1, 0, -1)$	$9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65, 69 \leq 74$
4	$(0, 0, 1, -1)$	$4, 34, 38, 46, 62, 94, 106, 122, 158, 166, 214, 218, 226, 274 \leq 277$
4	$(2, 0, 0, -1)$	≤ 100000
4	$(0, 2, 0, -1)$	$4, 9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65 \leq 68$
4	$(0, 0, 2, -1)$	≤ 100000
4	$(1, 1, 0, -1)$	≤ 100000
4	$(1, 0, 1, -1)$	$4, 8, 10, 14, 16, 20, 22, 25, 26, 28, 32, 34, 35, 38, 40, 44 \leq 45$
4	$(0, 1, 1, -1)$	≤ 100000
4	$(3, 0, 0, -1)$	$25 \leq 100000$
4	$(0, 3, 0, -1)$	$6, 9, 15, 18, 21, 25, 27, 33, 35, 39, 45, 49, 51, 54, 55, 57 \leq 62$
4	$(0, 0, 3, -1)$	$25 \leq 100000$
4	$(2, 1, 0, -1)$	$49 \leq 100000$
4	$(2, 0, 1, -1)$	≤ 100000
4	$(1, 2, 0, -1)$	$4 \leq 100000$
4	$(1, 0, 2, -1)$	≤ 100000
4	$(0, 2, 1, -1)$	$4 \leq 100000$
4	$(0, 1, 2, -1)$	$49 \leq 100000$
4	$(1, 1, 1, -1)$	$195, 897, 6213, 11285, 27889, 30745, 38503, 39601 \leq 100000$
4	$(4, 0, 0, -1)$	$4, 6, 25 \leq 100000$
4	$(0, 4, 0, -1)$	$4, 9, 15, 21, 25, 27, 28, 33, 35, 39, 45, 49, 51, 55, 57 \leq 62$
4	$(0, 0, 4, -1)$	$4, 6, 25 \leq 100000$
4	$(3, 1, 0, -1)$	≤ 100000
4	$(3, 0, 1, -1)$	$9, 27 \leq 100000$
4	$(1, 3, 0, -1)$	≤ 100000
4	$(1, 0, 3, -1)$	$9, 27 \leq 100000$
4	$(0, 3, 1, -1)$	≤ 100000
4	$(0, 1, 3, -1)$	≤ 100000

k	$(a_1, a_2, \ldots, a_k)$	pseudoprimes $\leq N$
4	$(2, 2, 0, -1)$	≤ 100000
4	$(2, 0, 2, -1)$	$285, 781, 1295, 1815, 5291, 23127, 23871, 59565 \leq 100000$
4	$(0, 2, 2, -1)$	≤ 100000
4	$(2, 1, 1, -1)$	≤ 100000
4	$(1, 2, 1, -1)$	$4, 8, 9, 16, 27, 32, 64, 81, 88, 128, 224, 243, 256, 351 \leq 500$
4	$(1, 1, 2, -1)$	≤ 100000

6. Questions

The entries in the table pose many questions.

Question 6.1. *Is the set of Lucas* $(0, 2, 1)$ *pseudoprimes equal to the set of Lucas* $(1, 0, 1, 1)$ *pseudoprimes?*

The pseudoprimes appearing in the Lucas $(0, 2, 1)$, $(1, 0, 1, 1)$, and $(0, 1, 2, 1)$ sequences are essentially the pseudoprimes appearing in the Lucas $(1, 1)$ pseudoprimes. Similarly, the pseudoprimes appearing in the Lucas $(1, 3, 1)$ and $(2, 0, 2, 1)$ sequences are essentially the pseudoprimes appearing in the Lucas $(2, 1)$ sequence.

The key to looking at Question 6.1 is the characteristic polynomials of these sequences.

Note that the characteristic polynomial of the Lucas $(1, 1)$ sequence is $x^2 - x - 1$. The characteristic polynomial of the Lucas $(0, 2, 1)$ sequence is

$$x^3 - 2x - 1 = (x^2 - x - 1)(x + 1). \tag{6.1}$$

The characteristic polynomial of the Lucas $(1, 0, 1, 1)$ sequence is

$$x^4 - x^3 - x - 1 = (x^2 - x - 1)(x^2 + 1). \tag{6.2}$$

And, the characteristic polynomial of the Lucas $(0, 1, 2, 1)$ sequence is

$$x^4 - x^2 - 2x - 1 = (x^2 - x - 1)(x^2 + x + 1). \tag{6.3}$$

Observe that in each case, the characteristic polynomial of the Lucas $(0, 2, 1)$, $(1, 0, 1, 1)$, or $(0, 1, 2, 1)$ sequence is equal to the characteristic polynomial of the Lucas $(1, 1)$ sequence multiplied by a cyclotomic polynomial [17].

Similarly, the characteristic polynomial of the Lucas $(2, 1)$ sequence is $x^2 - 2x - 1$. The characteristic polynomial of the Lucas $(1, 3, 1)$ is

$$x^3 - x^2 - 3x - 1 = (x^2 - 2x - 1)(x + 1). \tag{6.4}$$

And, the characteristic polynomial of the Lucas $(2, 0, 2, 1)$ sequence is

$$x^4 - 2x^3 - 2x - 1 = (x^2 - 2x - 1)(x^2 + 1). \tag{6.5}$$

Observe again that in each case, the characteristic polynomial of the Lucas $(1, 3, 1)$ or $(2, 0, 2, 1)$ sequence is equal to the characteristic polynomial of the Lucas $(2, 1)$ sequence multiplied by a cyclotomic polynomial.

Here is the theorem describing the situation above. The proof of the theorem will appear in a separate paper.

Theorem 6.1. *Suppose that the characteristic polynomial of a Lucas* $(a_1, a_2, \ldots, a_k = 1)$ *sequence is equal to the characteristic polynomial of a Lucas* $(b_1, b_2, \ldots, b_r = 1)$ *sequence multiplied by j polynomials, not necessarily distinct, each of which is a cyclotomic polynomial of order* m_1, m_2, ..., m_j, *respectively. Let* $m = lcm(m_1, m_2, \ldots, m_j)$. *Then the set of Lucas* $(b_1, b_2, \ldots, b_r = 1)$ *pseudoprimes relatively prime to m is equal to the set of Lucas* $(a_1, a_2, \ldots, a_k = 1)$ *pseudoprimes relatively prime to m.*

It was proved by Richard André-Jeannin [3] that the Lucas $(a_1, 1)$ sequence has an even Lucas $(a_1, 1)$ pseudoprime if and only if $a_1 \neq 1$. Richard André-Jeannin did not prove that the Lucas $(1, 1)$ sequence has no even Lucas $(1, 1)$ pseudoprime, but used results proving this which appeared in: D. J. White, J. N. Hunt, and L. A. G. Dresel [25], A. Di Porto [8], and P. S. Bruckman [7]. Since $x^2 + x + 1$ is a cyclotomic polynomial of order 3, it follows from (6.6) and Theorem 6.1 that the set of Lucas $(1, 1)$ pseudoprimes not divisible by 3 is equal to the set of Lucas $(0, 1, 2, 1)$ pseudoprimes not divisible by 3. In particular, 705, 24465, and 54705 are Lucas $(1, 1)$ pseudoprimes, but not Lucas $(0, 1, 2, 1)$ pseudoprimes. On page 129 of the book [19], Ribenboim notes that David Singmaster in 1983 found all Lucas $(1, 1)$ pseudoprimes $< 10^5$, and Ribenboim lists all 25 of these numbers on page 129. Since $x + 1$ and $x^2 + 1$ are cyclotomic polynomials of orders 2 and 4, respectively, it follows from (6.1), (6.2), Theorem 6.1, and Richard André-Jeannin's result that the set of odd Lucas $(0, 2, 1)$ pseudoprimes and odd Lucas $(1, 0, 1, 1)$ pseudoprimes are both equal to the set of all Lucas $(1, 1)$ pseudoprimes, since there are no even Lucas $(1, 1)$ pseudoprimes. This goes a long way towards answering Question 6.1 of whether the set of Lucas $(0, 2, 1)$ pseudoprimes is equal to the set of Lucas $(1, 0, 1, 1)$ pseudoprimes.

In fact, we can prove a stronger result regarding the Lucas $(0, 2, 1)$ and Lucas $(1, 0, 1, 1)$ pseudoprimes. Using the identities following the example sequences $G_n(0, 2, 1)$ and $G_n(1, 0, 1, 1)$, we can prove that the set of Lucas $(0, 2, 1)$ pseudoprimes is equal to the set of Lucas $(1, 0, 1, 1)$ pseudoprimes.

Similarly, it follows from (6.7), (6.8), Theorem 6.1, and André-Jeannin's result that the set of odd Lucas $(2, 1)$ pseudoprimes, odd Lucas $(1, 3, 1)$ pseudoprimes, and odd Lucas $(2, 0, 2, 1)$ pseudoprimes are all equal. Note that it follows from Proposition 2 in André-Jeannin's paper [3] that 2^k is a traditional Lucas $(2, 1)$ pseudoprime for all $k \geq 2$. However, 2^k is not a Lucas $(2, 1)$ pseudoprime for $k \geq 3$ by an argument similar to that given after Definition 5.1.

Another question is suggested by the table.

Question 6.2. *Is the set of Lucas* $(0, 2, 0, 1)$ *pseudoprimes equal to the set of Lucas* $(0, 4, 0, 1)$ *pseudoprimes?*

The following describes the situation for the Lucas $(0, a_2, 0, 1)$ pseudoprimes.

Theorem 6.2. *The Lucas* $(0, a_2, 0, 1)$ *pseudoprimes are precisely the odd composite natural numbers and the even integers* $2m \geq 4$ *for which* $m | G_m(a_2, 1)$.

Proof. It follows from the Newton formulas, the recursion relation defining the Lucas $(0, a_2, 0, 1)$ sequence, and by induction that

$$G_n = G_{-n} = 0 \qquad \text{if } n \geq 0 \text{ and } n \not\equiv 0 \pmod 2, \tag{6.6}$$

$$G_0 = 4, \tag{6.7}$$

$$G_2 = 2a_2, \tag{6.8}$$

$$G_{2(i+2)} = a_2 G_{2(i+1)} + G_{2i} \qquad \text{for } i \geq 0, \tag{6.9}$$

and

$$G_{-2i} = (-1)^i G_{2i} \qquad \text{for } i \geq 0. \tag{6.10}$$

In particular, we see by (6.6) that $G_1 = G_{-1} = 0$.

Consider the second-order Lucas sequence $\{G_n(a_2, 1)\}$. Then $G_0(a_2, 1) = 2$ and $G_1(a_2, 1) = a_2$. Note that

$$G_0(0, a_2, 0, 1) = 2G_0(a_2, 1) \text{ and}$$
$$G_2(0, a_2, 0, 1) = 2G_1(a_2, 1).$$

It now follows from (6.9) and the second-order recursion relation defining $\{G_n(a_2, 1)\}$ that

$$G_{2i}(0, a_2, 0, 1) = 2 \cdot G_i(a_2, 1)$$

for $i \geq 0$. The assertions concerning the Lucas $(0, a_2, 0, 1)$ pseudoprimes now follow from (6.6)-(6.10). □

The paper by Somer [22] gives comprehensive criteria for determining when $n | G_n(a_1, 1)$, which relates to Theorem 6.2.

The answer to Question 6.2 is no.

First of all, it is well-known that if $m | n$ and n/m is odd, then

$$G_m(a_1, 1) | G_n(a_1, 1).$$

Thus, if $6 | G_2(a_1, 1)$, then $6 | G_6(a_1, 1)$. It is also proven in Theorem 5 (v) of the reference by Somer [22] that if n is even and $n | G_n(a_1, 1)$, then $m | G_m(a_1, 1)$, when $m = G_n(a_1, 1)$.

Now consider the Lucas $(2, 1)$ sequence. Note that $6 | G_2(2, 1) = 6$. Thus, by our above discussion, $6 | G_6(2, 1) = 198$. Since 6 is even, it follows that

$$G_6(2, 1) = 198 | G_{198}(2, 1).$$

Hence, by Theorem 6.2, $2 \times 198 = 396$ is a Lucas $(0, 2, 0, 1)$ pseudoprime. By computation, 396 is not a Lucas $(0, 4, 0, 1)$ pseudoprime.

Next consider the Lucas $(4, 1)$ sequence. Note that $6 | G_2(4, 1) = 18$. Thus, it again follows by the arguments above that $6 | G_6(4, 1) = 5778$. Since 6 is even, it follows that

$$G_6(4, 1) = 5778 | G_{5778}(4, 1).$$

Hence, by Theorem 6.2, $2 \times 5778 = 11556$ is a Lucas $(0, 4, 0, 1)$ pseudoprime. But unfortunately, by computation, 11556 is also a Lucas $(0, 2, 0, 1)$ pseudoprime.

To find an example of a Lucas $(0, 4, 0, 1)$ pseudoprime that is not a Lucas $(0, 2, 0, 1)$ pseudoprime, we searched using the program in the Appendix. It turned out that the first composite even integers which are Lucas $(0, 2, 0, 1)$ pseudoprimes and not Lucas $(0, 4, 0, 1)$ pseudoprimes are 132, 396, and 1188. And the first composite even integer which is a Lucas $(0, 4, 0, 1)$ pseudoprime and not a Lucas $(0, 2, 0, 1)$ pseudoprime is 1284.

Another question is the following:

Question 6.3. *For every $k \geq 2$, is there a k-tuple $(a_1, a_2, \ldots, a_k = 1)$ such that there are an infinite number of Lucas $(a_1, a_2, \ldots, a_k = 1)$ pseudoprimes?*

The answer is yes. The reasoning goes as follows. In Theorem 2 of Rotkiewicz [20], it is proven that there are infinitely many Lucas $(a_1, 1)$ pseudoprimes. It now follows from Theorem 6.2 that for every $k \geq 2$, there is a k-tuple $(a_1, a_2, \ldots, a_k = 1)$ such that there are an infinite number of Lucas $(a_1, a_2, \ldots, a_k = 1)$ pseudoprimes.

There is a computational question we can ask.

Question 6.4. *Is there a Lucas $(0, 1, 1)$ pseudoprime?*

The answer is yes. In [14], Kurtz, Shanks, and Williams have found 55 Lucas $(0, 1, 1)$ pseudoprimes less than $50 \cdot 10^9$, the smallest of which is 27664033. Adams and Shanks [1] had previously found 14 of these pseudoprimes including the smallest one. It is of interest that 4 out of these 55 Lucas $(0, 1, 1)$ pseudoprimes are also Carmichael numbers. Grantham [11] has proved that there are infinitely many Lucas $(0, 1, 1)$ pseudoprimes. His method uses similar techniques as those employed in [2] to show that there are infinitely many Carmichael numbers, along with zero-density estimates for Hecke L-functions. In the paper [11], Grantham also proves that there are infinitely many Lucas $(a_1, a_2, \ldots, a_k = 1)$ psuedoprimes when the corresponding characteristic polynomial is square-free. This gives another affirmative answer to Question 6.3.

7. Appendix - C Program

```
// This C program finds generalized Lucas pseudoprimes.
// It uses the GNU MP library to handle the large integers.
// Input: Command Line
// Output: Screen
// gcc -o genlucaspsp genlucaspsp.c -lgmp
// genlucaspsp k max a_1 a_2 ... a_(k-1) a_k=+/-1
// k <= 1000
// The program outputs composite n between k and max satisfying
// g_n = a_1 mod n and h_n = -a_(k-1) mod n
// where g_0 = k, g_1 = a_1, g_2 = a_1 g_1 + 2a_2 , ...,
// and for n >= k, g_n = a_1 g_(n-1) + a_2 g_(n-2) + ... + a_k g_(n-k)
// and h_0 = k, h_1 = -a_(k-1)/a_k, h_2 = (-a_(k-1) h_1 -2a_(k-2))/a_k , ...
// and for n >= k, h_n = (-a_(k-1) h_(n-1) - a_(k-2) h_(n-2) - ...
// + h_(n-k))/a_k

#include <stdio.h>
#include <math.h>
#include <gmp.h>
```

```
main (int argc, char *argv[]) {
  mpz_t g[1001], h[1001], t, s, rg, rh;
  signed long int composite, n, i, j, k, max, a[1001];

  k = atoi (argv[1]);
  max = atoi (argv[2]);

  for (i=1; i<=k; i++) a[i] = atoi (argv[i+2]);

  for (i=0; i<=1000; i++)
  {
    mpz_init (g[i]);
    mpz_init (h[i]);
  }
  mpz_init (t);
  mpz_init (s);
  mpz_init (rg);
  mpz_init (rh);


  mpz_set_si (g[0],k);
  mpz_set_si (g[1],a[1]);
  i=2;
  while (i<k)
  {
    mpz_set_si (t,i);
    mpz_mul_si (t,t,a[i]);
    for (j=1; j<i; j++)
    {
      mpz_set_si (s,a[j]);
      mpz_mul (s,s,g[i-j]);
      mpz_add (t,t,s);
    }
    mpz_set (g[i],t);
    i++;
  }

  mpz_set_si (h[0],k);
  mpz_set_si (t,-a[k-1]);
  mpz_mul_si (t,t,a[k]);
  mpz_set (h[1],t);
  i=2;
  while (i<k)
  {
    mpz_set_si (t,-i);
    mpz_mul_si (t,t,a[k-i]);
    for (j=1; j<i; j++)
```

```
  {
    mpz_set_si (s,-a[k-j]);
    mpz_mul (s,s,h[i-j]);
    mpz_add (t,t,s);
  }
  mpz_mul_si (t,t,a[k]);
  mpz_set (h[i],t);
  i++;
}

n=k;

while ( n <= max )
{
  mpz_set_si (t,0);
  for (i=1; i<=k; i++)
  {
    mpz_set_si (s,a[i]);
    mpz_mul (s,s,g[k-i]);
    mpz_add (t,t,s);
  }
  mpz_set (g[k],t);

  mpz_set_si (t,0);
  for (i=1; i<k; i++)
  {
    mpz_set_si (s,-a[k-i]);
    mpz_mul (s,s,h[k-i]);
    mpz_add (t,t,s);
  }
  mpz_set (s,h[0]);
  mpz_add (t,t,s);
  mpz_mul_si (t,t,a[k]);
  mpz_set (h[k],t);

  mpz_sub_ui (t,g[k],a[1]);
  if (mpz_sgn(t)<0)
    mpz_mul_si (t,t,-1);
  mpz_mod_ui (rg,t,n);

  mpz_set_si (s,a[k-1]);
  mpz_mul_si (s,s,a[k]);
  mpz_add (t,h[k],s);
  if (mpz_sgn(t)<0)
    mpz_mul_si (t,t,-1);
  mpz_mod_ui (rh,t,n);
```

```
    if ((mpz_sgn( rg ) ==0) && (mpz_sgn( rh ) ==0))
    {
      composite = 0;
      if ((n!=2) && (n!=3))
      {
        if ((n%2)==0) composite = 1;
        if ((n%3)==0) composite = 1;
      }
      i=5;
      while (((i*i)<=n) && (composite==0))
      {
        if ((n%i)==0) composite = 1;
        i=i+2;
        if ((n%i)==0) composite = 1;
        i=i+4;
      }
      if (composite==1)
        printf("%d\n",n);
    }

    for (i=1; i<=k; i++) mpz_set (g[i-1],g[i]);
    for (i=1; i<=k; i++) mpz_set (h[i-1],h[i]);
    n++;
  }

  for (i=0; i<=1000; i++)
  {
    mpz_clear (g[i]);
    mpz_clear (h[i]);
  }
  mpz_clear (rg);
  mpz_clear (rh);
  mpz_clear (t);
  mpz_clear (s);

  exit(0);
}
```

References

[1] W. W. Adams and D. Shanks, "Strong Primality Tests that are not Sufficient," *Math. Comp.*, 39 (1982), 255–300.

[2] W. R. Alford, A. Granville, and C. Pomerance, "There are Infinitely Many Carmichael Numbers," *Ann. of Math.*, (2) 139 (1994), 703–722.

[3] R. André-Jeannin, "On the Existence of Even Fibonacci Pseudoprimes with Parameters P and Q," *The Fibonacci Quarterly*, 34 (1996), 75–78.

[4] A. O. L. Atkin, "Intelligent Primality Test Offer," *Computational Perspectives on Number Theory*, (Eds. D. A. Buell and J. T. Teitelbaum) AMS/IP Stud. Adv. Math. 7, Amer. Math. Soc., Providence, 1998, 1–11.
[5] N. G. W. H. Beeger, "On Even Numbers m Dividing $2^m - 2$," *Amer. Math. Monthly*, 58 (1951), 553–555.
[6] C. S. Bisht, "Some Congruence Properties of Generalized Lucas Integral Sequences," *The Fibonacci Quarterly*, 22 (1984), 290–295.
[7] P. S. Bruckman, "Lucas Pseudoprimes are Odd," *The Fibonacci Quarterly*, 32 (1994), 155–157.
[8] A. Di Porto, "Nonexistence of Even Fibonacci Pseudoprimes of the 1st Kind," *The Fibonacci Quarterly*, 31 (1993), 173–177.
[9] H. W. Gould, "The Girard-Waring Power Sum Formulas for Symmmetric Functions and Fibonacci Sequences," *The Fibonacci Quarterly*, 37 (1999), 135–140.
[10] J. Grantham, "Frobenius Pseudoprimes," *Math. Comp.*, 70 (2001), 837–891.
[11] J. Grantham, "There are Infinitely Many Perrin Pseudoprimes," Preprint.
[12] S. Gurak, "Pseudoprimes for Higher-Order Linear Recurrence Sequences," *Math. Comp.*, 55 (1980), 783–813.
[13] V. E. Hoggatt, Jr. and Marjorie Bicknell, "Some Congruences of the Fibonacci Numbers Modulo a Prime p," *Mathematics Magazine*, 47 (1974), 210–214.
[14] G. C. Kurtz, D. Shanks, and H. C. Williams, "Fast Primality Tests for Numbers Less Than $50 \cdot 10^9$," *Math. Comp.*, 46 (1986), 691–701.
[15] D. E. Littlewood, *A University Algebra*, William Heinemann, Ltd., London, 1958, p. 86.
[16] E. Lucas, "Théorie des Fonctions Numériques Simplement Périodiques," *Amer. J. of Math.*, 1 (1878), 197–240.
[17] T. Nagell, *Introduction to Number Theory*, Chelsea Publishing Company, 1981.
[18] R. Perrin, "Query 1484," *L'Intermédiaire des Mathématiciens*, 6 (1899), 76.
[19] P. Ribenboim, *The New Book of Prime Number Records*, Springer-Verlag, 2nd edition, New York, 1996.
[20] A. Rotkiewicz, "On the Pseudoprimes with Respect to the Lucas Sequence," *Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys.*, 21 (1973), 793–797.
[21] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, World-Wide Web URL www.research.att.com/∼njas/sequences/.
[22] L. Somer, "Divisibility of Terms in Lucas Sequences of the Second Kind by Their Subscripts," *Applications of Fibonacci Numbers*, Vol. 6 (Eds. G. E. Bergum et al.), Kluwer Academic Publishers, Dordrecht, 1996, pp. 473–486.
[23] G. Szekeres, "Higher Order Pseudoprimes in Primality Testing," *Combinatorics, Paul Erdős is Eighty, Bolyai Soc. Math. Stud.*, vol. 2, János Bolyai Math Soc., Budapest, 1996, 451–458.
[24] J. V. Uspensky, *Theory of Equations,* McGraw-Hill, New York, 1948.
[25] D. J. White, J. N. Hunt, and L. A. G. Dresel, "Uniform Huffman Sequences Do Not Exist," *Bull. London Math. Soc.*, 9 (1977), 193–198.

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