ON CONWAY'S RATS

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1. Introduction. John Conway invented a digital game called RATS [1], an acronym for Reverse, Add, Then Sort. A game of RATS produces a sequence of positive integers where the decimal digits of each positive integer are all nonzero and in nondecreasing order. The first number in a RATS sequence is a positive integer whose decimal digits are all nonzero and in nondecreasing order. For $i \geq 1$, to obtain the $(i+1)$st number in our RATS sequence, reverse the digits of the ith number, add this number to the ith number, delete the zero digits in the sum, and then sort the remaining digits of the sum in nondecreasing order. The resulting sequence of positive integers is the RATS sequence for the first number.

In this paper, we will examine many numerical sequences produced by the game of RATS. We will study the concept of cycling in RATS sequences. We will then try to prove some results about RATS sequences. Finally, we will discuss Conway's RATS conjecture and other questions.

2. RATS Sequences. Let us play the game of RATS starting with 3. The second element in this sequence is 6, since the reverse of 3 is 3 and $3 + 3 = 6$. The third element in this RATS sequence is 12 and the fourth element in this sequence is 33, since the reverse of 12 is 21 and $21 + 12 = 33$. The next elements in this sequence are 66 and 123, since the reverse of 66 is 66, $66 + 66 = 132$, and the sorted digits of 132 are 123. The next elements in this sequence are 444, 888, 1677, 3489, 12333, 44556, and 111. The entry 111 is produced since the reverse of 44556 is 65544, $65544 + 44556 = 11010$, and deleting the 0's in 110100 gives 111. Thus, the sequence produced by starting a game of RATS with 3 is

$$3,\ 6,\ 12,\ 33,\ 66,\ 123,\ 444,\ 888,\ 1677,\ 3489,$$
$$12333,\ 44556,\ 111,\ 222,\ 444,\ 888,\ 1677,\ 3489,$$
$$12333,\ 44556,\ 111,\ 222,\ 444,\ 888,\ 1677,\ 3489,$$
$$12333,\ 44556,\ 111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ \ldots.$$

This game results in a RATS cycle of length 8. Using the convention of stating the cycle by starting with the smallest number in the cycle, here 111, we have discovered the following RATS cycle.

$$111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556.$$

If we start the game of RATS with 1, the following sequence is generated.

1, 2, 4, 8, 16, 77, 145, 668, 1345, 6677, 13444, 55778,
133345, 666677, 1333444, 5567777, 12333445, 66666677,
133333444, 556667777, 1233334444, 5566667777,
12333334444, 55666667777, 123333334444, 556666667777,
1233333334444, 5566666667777, 12333333334444,
55666666667777, 123333333334444, 556666666667777,
1233333333334444, 5566666666667777,
12333333333334444, 55666666666667777,
123333333333334444, 556666666666667777, ...

To help describe this sequence, we will use the notation,

$$1_{m_1}\ 2_{m_2}\ \cdots\ 9_{m_9}.$$

This notation represents the number with m_1 1's followed by m_2 2's followed by $\cdots$ followed by m_9 9's. For convenience, we will omit the multiplicity factor m and the digit if the multiplicity factor is 0 and we will sometimes omit the multiplicity factor and just state the digit, if the multiplicity factor is 1. Again, the game of RATS starting with 1.

$$1,\ 2,\ 4,\ 8,\ 16,\ 7_2,\ 145,\ 6_28,\ 1345,\ 6_27_2,\ 134_3,\ 5_27_28,$$
$$13_345,\ 6_47_2,\ 13_34_3,\ 5_267_4,\ 123_34_25,\ 6_67_2,$$
$$13_54_3,\ 5_26_37_4,\ 123_44_4,\ 5_26_47_4,\ 123_54_4,\ 5_26_57_4,$$
$$123_64_4,\ 5_26_67_4,\ 123_74_4,\ 5_26_77_4,\ 123_84_4,\ 5_26_87_4,$$
$$123_94_4,\ 5_26_97_4,\ 123_{10}4_4,\ 5_26_{10}7_4,\ 123_{11}4_4,\ 5_26_{11}7_4,\ \ldots .$$

The last part of this sequence is called the divergent sequence.

Consider the RATS game starting with $16_{12}78$.

$$16_{12}78,\ 13_{12}49,\ 16_{11}78,\ 13_{11}49,\ 16_{10}78,\ 13_{10}49,$$
$$16_978,\ 13_949,\ 16_878,\ 13_849,\ 16_778,\ 13_749,\ 16_678,\ 13_649,$$
$$16_578,\ 13_549,\ 16_478,\ 13_449,\ 16_378,\ 13_349,\ 16_278,\ 13_249,$$
$$1678,\ 1349,\ 178,\ 149,\ 19,\ 1_2,\ 2_2,\ 4_2,\ 8_2,\ 167,\ 289,\ 1_227,$$
$$3_28_2,\ 1_22_3,\ 3_44,\ 6_37_2,\ 13_24_3,\ 5_27_4,\ 123_35_2,\ 6_57_2,$$
$$13_44_3,\ 5_26_27_4,\ 123_34_4,\ 5_26_37_4,\ 123_44_4,\ 5_26_47_4,$$
$$123_54_4,\ 5_26_57_4,\ 123_64_4,\ 5_26_67_4,\ 123_74_4, 5_26_77_4,$$
$$123_84_4,\ 5_26_84_4,\ 123_94_4,\ 5_26_97_4,\ 123_{10}4_4,\ 5_26_{10}7_4,\ \ldots .$$

This sequence follows a decreasing pattern and then enters the divergent sequence. We give some other RATS games.

$$7,\ 14,\ 5_2,\ 1_2,\ 2_2,\ 4_2,\ 8_2,\ 167,\ 289,\ 1_227,\ 3_28_2,$$
$$1_22_3,\ 3_44,\ 6_37_2,\ 13_24_3,\ 5_27_4,\ 123_35_2,\ 6_57_2,$$
$$13_44_3,\ 5_26_27_4,\ 123_34_4,\ 5_26_37_4,\ 123_44_4,\ 5_26_47_4,$$
$$123_54_4,\ 5_26_57_4,\ 123_64_4,\ 5_26_67_4,\ 123_74_4,\ 5_26_77_4,\ \ldots .$$

$$9,\ 18,\ 9_2,\ 189,\ 1_27,\ 28_2,\ 1_27,\ 28_2,\ 1_27,\ 28_2, \ldots .$$

$$12,\ 3_2,\ 6_2,\ 123,\ 4_3,\ 8_3,\ 167_2,\ 3489,$$
$$123_3,\ 4_25_26,\ 1_3,\ 2_3,\ 4_4,\ 8_3,\ 167_2,\ 3489,$$
$$123_3,\ 4_25_26,\ 1_3,\ 2_3,\ 4_4,\ 8_3,\ 167_2,\ 3489,$$
$$123_3,\ 4_25_26,\ 1_3,\ 2_3,\ 4_4,\ 8_3,\ 167_2,\ 3489,\ \ldots .$$

$$69,\ 156,\ 78,\ 156,\ 78,\ 156,\ 78,\ 156,\ 78,\ 156,\ \ldots .$$

$28_5,\ 1_27_4,\ 458_39,\ 134_37_2,\ 8_39_2,\ 17_28_3,\ 156_39,\ 1_22_23_2,$

$4_6,\ 8_6,\ 167_5,\ 345_389,\ 1_423_3,\ 3_24_6,\ 7_48_4,\ 156_7,\ 123_578,$

$6_59_4,\ 1356_7,\ 1234_4789,\ 1_56_27,\ 2_27_48_2,\ 1_45_4,\ 6_8,\ 123_7,$

$4_25_26_5,\ 1_42_43,\ 3_64_3,\ 6_37_6,\ 134_55_3,\ 6_28_69_2,\ 156_37_6,$

$234_6589,\ 1_32_28_49_2,\ 1_87,\ 2_78_2,\ 1_34_45,\ 5_46_28_2,\ 12_434_3,$

$45_46_4,\ 1_32_6,\ 3_64_3,\ \ldots .$

All of these RATS sequences are fascinating and complex and have many unanswered questions associated with them. The divergent sequence given earlier will be of particular interest when we discuss Conway's RATS conjecture. Next, we will explore RATS sequences which eventually cycle.

3. Known Cycles. Let

$$\{x_i\}_{i=1}^{\infty}$$

be a RATS sequence. A RATS cycle is a finite subsequence of a RATS sequence,

$$\{x_i\}_{i=s}^{p-1},$$

such that $x_s < x_i$ for all $s < i < p$, and for all $i \geq s$, $x_i = x_{i+p}$. In addition, p is the smallest integer with this property. The length of a RATS cycle is p.

The following are RATS cycles of various lengths.

Length 2

$$1_327,\ 23_28_2\ .$$

Length 6

$$1_42_65_2,\ 3_44_46_4,\ 8_49_8,\ 178_79_4,\ 1_27_78_4,\ 15_66_39_2\ .$$

Length 9

$$1_{27},\ 2_{27},\ 4_{27},\ 8_{27},\ 167_{26},\ 345_{24}89,$$

$$1_{25}23_3,\ 2_{21}3_24_6,\ 4_{13}5_46_{12}\ .$$

Length 12

$$12_{21}3,\ 4_{23},\ 8_{23},\ 167_{22},\ 345_{20}89,\ 1_{21}23_3,$$

$$2_{17}3_24_6,\ 4_95_46_{12},\ 1_{19}2_6,\ 2_{13}3_{12},\ 45_{24},\ 1_{23}9\ .$$

Some RATS cycles can be grouped together because they have a common form. We will use the term family of RATS cycles. Some of these families can be described with a positive or nonnegative integer n.

Length 3

$$1_{6n}3_n,\ 2_{5n}4_{2n},\ 4_{3n}6_{4n}\ .$$

Length 4

$$1_{4n+6}7_{11n+15},\ 45_{7n+8}8_{8n+11}9,\ 134_{14n+17}7_{n+3},\ 1_2 2_{2n+1}8_{13n+16}9_2\ .$$

Length 10

$$1_{325n+158}2_{16n+8},\ 2_{309n+150}3_{32n+16},\ 4_{277n+134}5_{64n+32},$$
$$8_{213n+102}9_{128n+64},\ 17_{85n+39}8_{256n+127},\ 156_{47n+77}7_{171n+87}9,$$
$$1_2 3_2 4_{340n+154}5_{n+9},\ 6_4 8_{339n+153}9_{2n+10},\ 156_7 7_{337n+147}8_{4n+12},$$
$$1345_{333n+158}6_{8n+6}9\ .$$

Curt McMullen has discovered these and many other RATS cycles. His results can be found in [2]. In all, he found 25 RATS cycles and 7 families of RATS cycles. Some other highlights from his results include a RATS cycle of length 9 starting with 1_{27}, a RATS cycle of length 18 starting with 1_{26}, and a RATS cycle of length 24 starting at $1_2 4_{48} 5_{46}$.

4. Programs and New Cycles. At this point, we set out to discover some new cycles.

First, we wrote a Pascal program to input from the keyboard the 1st number of a RATS game. The input is a sequence of nonzero digits in nondecreasing order, interspersed with their multiplicities. The input is terminated by ctrl-z. We then output to the screen the RATS sequence associated with the input. For example, if we input

1 2 2 3 3 4 ctrl-z

to denote that we want to play the RATS game starting with $112223333 = 1_2 2_3 3_4$, the RATS sequence output is

1_2 2_3 3_4, 4_5 5_4, 8_1 9_8, 1_1 7_1 8_1 9_7,

1_2 7_2 8_2 9_4, 1_4 7_6, 4_1 5_1 8_7 9_1,

1_1 3_1 4_3 7_6, 1_1 3_1 4_3 7_6,

1_1 2_6 5_1 8_1 9_1, 1_3 4_4 7_1 8_1,

5_2 8_5 9_2, 1_1 4_1 5_3 7_5,

1_1 2_1 3_6 8_1 9_1, 1_3 6_5 7_1,

2_1 3_2 7_3 8_3, 1_2 2_4 5_3, 4_3 6_4 7_2, ...

The RATS sequence we generate will continue indefinitely until we enter

ctrl-c.

This Pascal program can be found on the WWW at

http://153.91.1.112/~curtisc/rats/prog1.txt

We wrote a 2nd Pascal program to help us search for RATS cycles. Our idea was to look for RATS cycles in a particular region. We decided to concentrate on RATS games starting with $1_m 2_n$, where m and n are positive integers. We picked this region because many of the cycles discovered by Curt McMullen have entries of this form. Thus far, we have explored the region where

$$1 \leq m \leq 4010 \quad \text{and} \quad 1 \leq n \leq 4010.$$

We accomplished this search in phases; first we would search a region

$$1 \leq m \leq i \quad \text{and} \quad 1 \leq n \leq i$$

and then we would choose $j \geq i$ and search the region

$$1 \leq m \leq j \quad \text{and} \quad 1 \leq n \leq j, \qquad \text{where } m \geq i \text{ or } n \geq i.$$

We searched the region by printing to the screen and outputting to a file the elements which produced RATS sequences which didn't degenerate into the divergent sequence and after x elements in the sequence, the sequence entries had more than y digits. The x and y values were arbitrary and based on heuristics which we developed by studying the game of RATS. Toward the outer edge of this region we chose $x = 250$ and $y = 2500$. If a value was output by our program, we checked the sequence generated by this RATS game using our first program. We found both new RATS cycles and new families of RATS cycles. This Pascal program can be found on the WWW at

http://153.91.1.112/~curtisc/rats/prog2.txt

All of McMullen's and our material are tabulated and can be found on the WWW at

http://153.91.1.112/~curtisc/rats/ratscycles.txt

This table contains a total of 178 RATS cycles and 42 families of RATS cycles.

Some highlights of the table are given below.

1. We found RATS cycles of length 13, 15, 16, 17, 19, 20, 22, 29, 30, 32, 34, and 36. The RATS cycles we found of length 29, 30, 34, and 36 begin with $1_2 2_{424} 3_{3591} 5_{10}$, $1_{3309} 2_{2600}$, $1_{16} 2_{760} 3_{183}$, and $1_6 2_{281} 3_{4726} 8_2$, respectively.

2. 2. We found a family of RATS cycles of length 11, 14, 16, and 18. The family of RATS cycles of length 14 starts at $1_{2957n+73} 2_{2504n+38} 3_2 4_4$.

3. We discovered two families of RATS cycles of the form

$$1_2 2_{10} 3_{634n+296} 4_{276n+156}$$

and

$$1_2 2_{10} 3_{951n+296} 4_{414n+156}.$$

Both families are of length 12.

4. We found another interesting family of RATS cycles.

$$1_{128n+133} 2_{128n+119} 3_4 6_{16n+15} 7.$$

This family is of length 4.

5. Theorems, Open Questions, and the RATS Conjecture. We now present some theorems about the game of RATS.

Theorem 1. There are no RATS cycles of length 1.

Proof. By contradiction. Suppose

$$\{x\}_{i=1}^{\infty}$$

is a RATS cycle of length 1. Let s be the smallest digit of x and l be the largest digit of x. If $s \geq 2$, then $l = 2$ produces a contradiction as does $l = 3, 4, 5, 6, 7, 8, 9$. Therefore, $s = 1$. Also, if $l \leq 8$, then $s = 8$ produces a contradiction as does $s = 7, 6, 5, 4, 3, 2, 1$. Therefore, $l = 9$.

Thus,

$$x = 1_{m_1} 2_{m_2} 3_{m_3} 4_{m_4} 5_{m_5} 6_{m_6} 7_{m_7} 8_{m_8} 9_{m_9},$$

and $m_1 > 0$ and $m_9 > 0$. But then it can be shown that if $m_1 \geq 1$ and $m_9 \geq 1$, then $m_1 \geq 2$ and $m_9 \geq 2$. Similarly, it can be shown that for any $k \geq 1$, if $m_1 \geq k$ and $m_9 \geq k$, then $m_1 \geq k+1$ and $m_9 \geq k+1$,. But this is a contradiction. Therefore, Theorem 1 is proved.

Theorem 2. Let

$$\{x_i\}_{i=1}^{\infty}$$

be a RATS sequence. Then either

$$x_i \equiv 0 \pmod 3, \quad \text{for all } i \geq 1$$

or

$$x_i \not\equiv 0 \pmod 3, \quad \text{for all } i \geq 1.$$

The proof of Theorem 2 follows directly from the RATS process.

As more sequences are studied, perhaps more theorems can be invented. Here are some questions which can be studied.

1. Are there RATS cycles of length 21, 23, 25, etc?
2. Are there arbitrarily long RATS cycles?
3. Is there a RATS cycle in which some sequence entry contains every digit 1, 2, 3, 4, 5, 6, 7, 8, and 9?

Finally, John Conway has a simple sounding, yet tremendously hard conjecture based on his RATS game. So far, every number with nonzero digits in nondecreasing order (up to 15 digits) which start a RATS game either gets into a cycle or enters the divergent sequence. Conway's RATS conjecture is that this is true for every number with nonzero digits in nondecreasing order. This is still an open problem.

References

1. J. Conway, "Play it again ... and again ...," *Quantum,* November/December 1990, 30–31,63.
2. R. K. Guy, "Conway's RATS and Other Reversals," *The American Mathematical Monthly,* 96 (1989), 425–428.