BASE 10 RATS CYCLES AND ARBITRARILY LONG BASE 10 RATS CYCLES

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1. Introduction. John Conway invented a digital game called RATS [1], an acronym for Reverse, Add, Then Sort. A game of RATS produces a sequence of positive integers. The decimal digits of each positive integer in the sequence are all nonzero and in nondecreasing order. The first number in a RATS sequence is a positive integer whose decimal digits are all nonzero and in nondecreasing order. To produce the $(i+1)$st number in a RATS sequence from the i number, for $i \geq 1$, reverse the digits of the ith number, add this number to the ith number, delete the zero digits in the sum, and then sort the remaining digits of the sum in nondecreasing order. The resulting sequence of positive integers is the RATS sequence for the first number.

For example, if we begin a game of RATS with 3, then the RATS sequence is

$$
\begin{array}{l}
3,\ 6,\ 12,\ 33,\ 66,\ 123,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556, \\
111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556, \\
111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556, \\
111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556, \\
\vdots
\end{array}
$$

Note that 6 follows 3 in the game since $3+3=6$, 123 follows 66, since $66+66=132$ and then sorting the digits 132 results in the number 123, and 111 follows 44556 in the sequence, since $44556+65544=110100$ and deleting the 0's leaves 111. It is clear that the last 8 numbers cycle. This is called a RATS cycle. The length of the cycle is 8 and the smallest number in the cycle is 111.

Another interesting game starts with 1. The resulting RATS sequence is

1, 2, 4, 8, 16, 77, 145, 668, 1345, 6677, 13444, 55778, 133345,

666677, 1333444, 5567777, 12333445, 66666677, 133333444, 556667777,

12333334444, 55666667777,

123333334444, 556666667777,

1233333334444, 5566666667777,

12333333334444, 55666666667777,

123333333334444, 556666666667777,

1233333333334444, 5566666666667777,

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123333333333333334444, 556666666666666667777,

1233333333333333334444, 5566666666666666667777,

12333333333333333334444, 55666666666666666667777,

123333333333333333334444, 556666666666666666667777,

1233333333333333333334444, 5566666666666666666667777,

⋮

This recurring pattern is known as Conway's divergent sequence.

2. Base 10 RATS Cycles. In [3], Curt McMullen gave a list of the base 10 RATS cycles he had discovered. We set out to search heuristically for some more base 10 RATS cycles. In [2], we compiled a list of the base 10 RATS cycles which included McMullen's and some new ones we found. Here, we will present some additional base 10 RATS cycles.

Due to the size and repetitive nature of the positive integers in our base 10 RATS games, we will use superscripts to denote repeated digits in a number. For example, the number 5566666666666667777 will be represented by

$$5^2 6^{11} 7^4.$$

We had noticed that a number of the known base 10 RATS cycles contained positive integers consisting of 1's and 2's. Therefore, we decided to look for base 10 RATS cycles in this region. That is, we searched for base 10 RATS cycles by starting RATS games with numbers of the form

$$1^x 2^y.$$

To automate our search as much as possible and because of the number of digits in some of our base 10 RATS games, we wrote several C programs to play our base 10 RATS games and search for base 10 RATS cycles. All of these programs can be found on the World-Wide Web at

http://www.math-cs.cmsu.edu/~curtisc/rats/

Here are some highlights of what we found. We found base 10 RATS cycles of length 13, 15, 16, 17, 19, 20, 21, 22, 25, 26, 29, 30, 32, 34, 36, 40, 45, and 69. McMullen had not found any base 10 RATS cycles of these lengths. One of the new ones of length 13 is

$$1^{812}2^{1448},\ 3^{1624}4^{636},\ 6^{988}7^{1272},$$
$$134^{1975}5^{284},\ 6^2 8^{1695}9^{564},\ 156^3 7^{1133}8^{1124},$$
$$1345^{20}6^{2238}9,\ 1^4 2^{40}3^{2216},\ 4^8 5^{80}6^{2172},$$
$$1^{16}2^{160}3^{2084},\ 4^{32}5^{320}6^{1908},\ 1^{64}2^{640}3^{1556},\ 4^{128}5^{1280}6^{852}.$$

We also found additional RATS cycles of lengths 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18, and 24. One of the new ones of length 8 is

$$1^6 5^2 7^8,\ 23^3 8^{11}9,\ 1^2 2^7 7^8,\ 48^4 9^{12},$$
$$1348^8 9^7,\ 1^2 3^2 4^2 7^4 8^8,\ 1^2 2^3 3^4 5^2 6^4 9^2,\ 1^3 68^9 9^3.$$

Some of the base 10 RATS cycles we found of length 36, 45, and 69 begin with

$$1^{6}2^{281}3^{4726}8^{2},$$

$$1^{6354}2^{4140},$$

$$\text{and } 1^{6780}2^{6814}3^{2}4^{4},$$

respectively. In particular, the base 10 RATS cycle of length 36 is

$$1^{6}2^{281}3^{4726}8^{2},\ 4^{8}5^{562}6^{4441}9^{4},\ 1^{9}2^{1124}3^{3876}4^{7},$$

$$4^{4}5^{2262}6^{2750},\ 1^{8}2^{4524}3^{484},\ 4^{4064}5^{952},$$

$$8^{3112}9^{1904},\ 17^{1209}8^{3807},\ 156^{2417}7^{2597}9,$$

$$1^{2}3^{2}4^{4834}5^{179},\ 6^{4}8^{4663}9^{350},\ 156^{7}7^{4317}8^{692},$$

$$1345^{3648}6^{1366}9,\ 1^{2288}2^{2728},\ 3^{4576}4^{440},$$

$$6^{4136}7^{880},\ 13^{3257}4^{1759},\ 5^{2}6^{1499}7^{3516},$$

$$123^{3}4^{2998}5^{2015},\ 6^{2}7^{2}8^{994}9^{4020},\ 156^{3}7^{4}8^{1988}9^{3022},$$

$$1^{2}5^{2}6^{6}7^{8}8^{3976}9^{1025},\ 1^{4}5^{4}6^{12}7^{2985}8^{2014}, 134^{7}5^{1015}6^{3988}9^{4},$$

$$1^{12}2^{2030}3^{2970}4^{4},\ 4^{16}5^{4068}6^{932},\ 1^{3184}2^{1832},$$

$$2^{1352}3^{3664},\ 5^{2704}6^{2312},\ 1^{394}2^{4623},$$

$$3^{788}4^{4229},\ 7^{1576}8^{3441},\ 156^{3151}7^{1865},$$

$$23^{1289}4^{3726}89,\ 1^{3}2^{2}7^{2575}8^{2439},\ 1^{4}5^{141}6^{4868}9^{3}.$$

We found 3 base 10 RATS cycles of length 32. These RATS cycles begin with

$$1^{6235}2^{106},$$

$$1^{2110}2^{878}3^{2}4^{4},$$

$$\text{and } 1^{2731}2^{2838}3^{2}4^{2}.$$

We found families of base 10 RATS cycles of length 11, 13, 14, 16, and 18. One family of RATS cycles of length 14 starts at

$$1^{2957n+73}2^{2504n+38}3^{2}4^{4}.$$

The proof that $1^{2957n+73}2^{2504n+38}3^2 4^4$ is the least member of a family of RATS cycle of length 14 follows from the RATS cycle

$$1^{2957n+73}2^{2504n+38}3^2 4^4,$$

$$2^{453n+29}3^{5008n+76}4^4 5^8,$$

$$5^{906n+34}6^{4555n+67}7^{16},$$

$$12^{1812n+37}3^{3649n+80},$$

$$4^2 5^{3624n+74}6^{1837n+42},$$

$$1^{1787n+38}2^{3674n+80},$$

$$3^{3574n+76}4^{1887n+42},$$

$$6^{1687n+34}7^{3774n+84},$$

$$134^{3374n+67}5^{2087n+50},$$

$$6^2 8^{1287n+21}9^{4174n+96},$$

$$156^3 8^{2574n+42}9^{2887n+73},$$

$$1^2 5^2 6^6 8^{5148n+84}9^{313n+26},$$

$$1^4 5^4 6^{12} 7^{4835n+68}8^{626n+32},$$

$$134^7 5^{4209n+80}6^{1252n+24}9^4.$$

We also found some new smaller families of RATS cycles. One of length 8 starts with $1^{164n+110}7^{91n+61}$. The proof that this is a family of RATS cycles of length 8 follows from the RATS cycle

$$1^{164n+110}7^{91n+61},\ 2^{73n+49}8^{182n+122},$$

$$1^{146n+98}7^{109n+73},\ 2^{37n+25}8^{218n+146},$$

$$1^{74n+50}7^{181n+121},\ 45^{107n+70}8^{148n+99}9,$$

$$134^{214n+141}7^{41n+29},\ 1^2 2^{82n+53}8^{173n+114}9^2.$$

We found

$$1^{128n+133}2^{128n+119}3^4 6^{16n+15}7$$

to be an interesting family of base 10 RATS cycles of length 4. The proof that this is a family of base 10 RATS cycles of length 4 follows from the RATS cycle

$$1^{128n+133}2^{128n+119}3^4 6^{16n+15}7,$$

$$3^{224n+226}4^{16n+14}7^{32n+30}8^2,$$

$$1^{64n+61}2^3 6^{176n+180}7^{32n+27}8,$$

$$23^{144n+143}7^{71n+59}8^{64n+59}9^3.$$

The entire list of 230 base 10 RATS cycles and 86 base 10 RATS families we know to date can be found on the WWW at

http://www.math-cs.cmsu.edu/˜curtisc/rats/ratscycles.txt .

An excerpt of the beginning of the table follows. To save space in the table and to represent superscripts in computer output, we will use the notation x^y or x^{ y } to denote x^y.

Base 10 RATS Cycles

Length	Least Member
2	1^5 2^3 67
2	1^3 27
2	1^2 6^2 7
2	1^{2n} 7^n
2	78
3	1^{6n} 3^n
4	1^{128n + 133} 2^{128n + 119} 3^4 6^{16n + 15} 7
4	1^{17} 2^{15} 67
4	1^9 2^3 678
4	1^9 2^7 7
4	1^{21} 2^3 7^3 8^3
4	1^3 4^6 5^3 6
4	1^8 6^8 7^4
4	1^{4n + 6} 7^{11n + 15}
5	1^2 2^6 3^{58} 4^{20n + 200} 5^{11n + 265}
5	1^{24n} 3^{7n}

3. Arbitrarily Long Base 10 RATS Cycles. In [4], Cooper and Kennedy asked if there are arbitrarily long base 10 RATS cycles. Our next goal is to answer that question in the affirmative.

Lemma 1. Let $t \geq 3$ be an odd integer. Then

$$
\begin{aligned}
&1^{6\cdot 2^{t-3}}3^{2\cdot 2^{t-3}-1},\\
&2^{4\cdot 2^{t-3}+1}4^{4\cdot 2^{t-3}-2},\\
&4^{3}6^{8\cdot 2^{t-3}-4},\\
&1^{6\cdot 2^{0}}3^{8\cdot 2^{t-3}-6\cdot 2^{0}-1},\\
&4^{6\cdot 2^{1}}6^{8\cdot 2^{t-3}-6\cdot 2^{1}-1},\\
&1^{6\cdot 2^{2}}3^{8\cdot 2^{t-3}-6\cdot 2^{2}-1},\\
&4^{6\cdot 2^{3}}6^{8\cdot 2^{t-3}-6\cdot 2^{3}-1},\\
&1^{6\cdot 2^{4}}3^{8\cdot 2^{t-3}-6\cdot 2^{4}-1},\\
&4^{6\cdot 2^{5}}6^{8\cdot 2^{t-3}-6\cdot 2^{5}-1},\\
&\vdots\\
&1^{6\cdot 2^{t-5}}3^{8\cdot 2^{t-3}-6\cdot 2^{t-5}-1},\\
&4^{6\cdot 2^{t-4}}6^{8\cdot 2^{t-3}-6\cdot 2^{t-4}-1}
\end{aligned}
$$

is a base 10 RATS cycle of length t.

Lemma 2. Let $t \geq 8$ be an even integer. Then

$$
\begin{aligned}
&1^{216\cdot 2^{t-8}}3^{40\cdot 2^{t-8}-1},\\
&2^{176\cdot 2^{t-8}+1}4^{80\cdot 2^{t-8}-2},\\
&4^{96\cdot 2^{t-8}+3}6^{160\cdot 2^{t-8}-4},\\
&1^{192\cdot 2^{t-8}+6}3^{64\cdot 2^{t-8}-7},\\
&2^{128\cdot 2^{t-8}+13}4^{128\cdot 2^{t-8}-14},\\
&4^{27}6^{256\cdot 2^{t-8}-28},\\
&1^{54}3^{256\cdot 2^{t-8}-55},\\
&4^{108}6^{256\cdot 2^{t-8}-109},\\
&1^{216\cdot 2^{0}}3^{256\cdot 2^{t-8}-216\cdot 2^{0}-1},\\
&4^{216\cdot 2^{1}}6^{256\cdot 2^{t-8}-216\cdot 2^{1}-1},\\
&1^{216\cdot 2^{2}}3^{256\cdot 2^{t-8}-216\cdot 2^{2}-1},\\
&4^{216\cdot 2^{3}}6^{256\cdot 2^{t-8}-216\cdot 2^{3}-1},\\
&1^{216\cdot 2^{4}}3^{256\cdot 2^{t-8}-216\cdot 2^{4}-1},\\
&4^{216\cdot 2^{5}}6^{256\cdot 2^{t-8}-216\cdot 2^{5}-1},\\
&\vdots\\
&1^{216\cdot 2^{t-10}}3^{256\cdot 2^{t-8}-216\cdot 2^{t-10}-1},\\
&4^{216\cdot 2^{t-9}}6^{256\cdot 2^{t-8}-216\cdot 2^{t-9}-1}
\end{aligned}
$$

is a base 10 RATS cycle of length t.

In addition, let n be a positive integer. Then, for any fixed odd integer $t \geq 3$, there is a base 10 RATS family of length t with smallest element

$$1^{(6\cdot 2^{t-3})n}3^{(2\cdot 2^{t-3}-1)n}$$

and for any fixed even integer $t \geq 8$, there is a base 10 RATS family of length t with smallest element

$$1^{(216\cdot 2^{t-8})n}3^{(40\cdot 2^{t-8}-1)n}.$$

We now have the following theorem.

Theorem. Let $t \geq 2$ be a positive integer. Then there exists a base 10 RATS cycle of length t.

Proof. The cases where $t \geq 3$ is an odd integer and $t \geq 8$ is an even integer are shown by Lemmas 1 and 2. The remaining cases can be handled by the following table.

Cycle of Length 2	$117,\ 288.$
Cycle of Length 4	$1^6 7^{15},\ 45^8 8^{11} 9,\ 134^{17} 7^3,\ 1^2 28^{16} 9^2.$
Cycle of Length 6	$1^4 2^6 5^2,\ 3^4 4^4 6^4,\ 8^4 9^8,\ 178^7 9^4,\ 1^2 7^7 8^4,\ 15^6 6^3 9^2.$

This completes the proof.

We also found two other interesting collections of RATS cycles. They can be stated as follows.

Lemma 3. Let t be a positive integer. Then

$$
\begin{aligned}
&1^{2^{2t-1}+1} 2^{2^{2t-1}-1} 7,\\
&23^{4^t-2} 8^2,\\
&1^{2^1+1} 2^{2^1-1} 6^{4^t-4^1} 7,\\
&23^{4^t-2\cdot 4^1} 7^{4^1} 8^{4^1-1} 9,\\
&1^{2^3+1} 2^{2^3-1} 6^{4^t-4^2} 7,\\
&23^{4^t-2\cdot 4^2} 7^{4^2} 8^{4^2-1} 9,\\
&1^{2^5+1} 2^{2^5-1} 6^{4^t-4^3} 7,\\
&23^{4^t-2\cdot 4^3} 7^{4^3} 8^{4^3-1} 9,\\
&\vdots\\
&1^{2^{2t-3}+1} 2^{2^{2t-3}-1} 6^{4^t-4^{t-1}} 7,\\
&23^{4^t-2\cdot 4^{t-1}} 7^{4^{t-1}} 8^{4^{t-1}-1} 9
\end{aligned}
$$

is a base 10 RATS cycle of length $2t$.

Lemma 4. Let t be a positive integer. Then

$$1^{4^t+1}2^{4^t-1}67,$$
$$3^{2\cdot 4^t-2}7^2 8^2,$$
$$1^{4^1+1}2^{4^1-1}6^{2\cdot 4^t-2\cdot 4^1+1}7,$$
$$23^{2\cdot 4^t-4^2+1}7^{2\cdot 4^1}8^{2\cdot 4^1-1}9,$$
$$1^{4^2+1}2^{4^2-1}6^{2\cdot 4^t-2\cdot 4^2+1}7,$$
$$23^{2\cdot 4^t-4^3+1}7^{2\cdot 4^2}8^{2\cdot 4^2-1}9,$$
$$1^{4^3+1}2^{4^3-1}6^{2\cdot 4^t-2\cdot 4^3+1}7,$$
$$23^{2\cdot 4^t-4^4+1}7^{2\cdot 4^3}8^{2\cdot 4^3-1}9,$$
$$\vdots$$
$$1^{4^{t-1}+1}2^{4^{t-1}-1}67,$$
$$23^{2\cdot 4^t-4^t+1}7^{2\cdot 4^{t-1}}8^{2\cdot 4^{t-1}-1}9$$

is a base 10 RATS cycle of length $2t$.

4. Questions. We conclude this article with several open questions. One problem would be to find other base 10 RATS cycles and base 10 families of RATS cycles. In addition, we could study the game of RATS in other bases and find RATS cycles and families of RATS cycles in these bases. Finally, John Conway has a simple sounding, yet tremendously hard conjecture based on his base 10 RATS game. So far, every number with nonzero digits in nondecreasing order (up to 15 digits) which starts a base 10 RATS game either gets into a cycle or enters the divergent sequence. Conway's RATS conjecture is that this is true for every number with nonzero digits in nondecreasing order.

References

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