

ON A SEATING REARRANGEMENT PROBLEM

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1. **Introduction.** In [1], the following problem was presented.

A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). Determine whether or not this directive can be carried out.

In [2], we showed that there is no way that this directive can be carried out. To see this, consider a desk grid as a 5×5 checkerboard of X and O squares with an X in the upper left corner.

X	O	X	O	X
O	X	O	X	O
X	O	X	O	X
O	X	O	X	O
X	O	X	O	X

Now suppose we could carry out this directive. Then the pupils in the X squares must move to the O squares. But, since there are more X squares than O squares, the pigeonhole principle [3] says that two pupils in X squares must move to the same O square, a contradiction. In fact, there is no way to carry out this directive if the classroom has an odd number of rows with an odd number of desks per row.

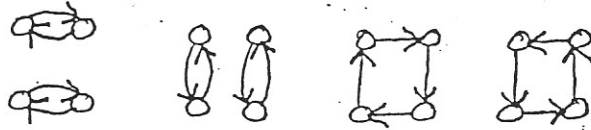
Also in [2], we became interested in the following variation to the problem.

A classroom has $2m$ rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or to the one on his right (of course, not all these options are possible for all students). How many ways can this directive be carried out?

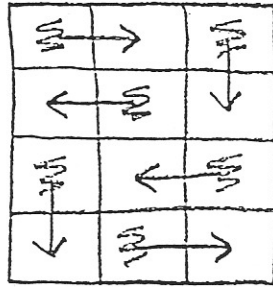
We began an attack on this problem by attempting to solve the $2 \times n$ problem. To do this, we first defined the concept of a $2 \times n$ seating rearrangement. To count the number of $2 \times n$ seating rearrangements, we partitioned the seating rearrangements into four disjoint subsets (depending on how the pupils in the last column move). Next, we derived equations between the number of $2 \times (n+1)$ seating rearrangements and the number of $2 \times n$ seating rearrangements. Specifically, using the idea of 1-1 correspondence, we found mappings between each of the four disjoint subsets of $2 \times (n+1)$ seating rearrangements and the previous four disjoint subsets of $2 \times n$ seating rearrangements. Solving this system of recurrence relations with its initial conditions, we discovered that the number of seating rearrangements in a classroom with 2 rows and n desks per row is F_{n+1}^2 , where F_n denotes the n th Fibonacci number! However, extending these methods in general from 2 to $2m$ rows seemed hard. Even for a particular $m > 1$, the number of partitions of the $2m \times n$ seating rearrangements and their interrelationships appeared to be large and complicated.

Therefore, a new approach was needed. The techniques and results we will use to solve this problem are similar to those used in counting the number of non-overlapping dimer coverings on a $2m \times 2n$ lattice with sharp boundaries [4]. However, we must modify some of the methods to suit our particular situation. The details will be included because 1) the methods are not well known and 2) [4] is out of print! Finally, we will present some interesting and surprising corollaries.

2. **Notation and Square Result.** Let $S(2m, n)$ denote the number of seating rearrangements for $2m$ rows with n desks per row. For example, $S(2, 2) = 4$ since either the horizontal neighbors exchange seats, the vertical neighbors exchange seats, or the pupils exchange seats clockwise or counterclockwise. This situation is pictured below.



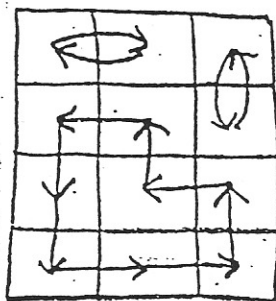
Now consider a $2m \times n$ checkerboard superimposed over the $2m \times n$ desk array with a black square located in the upper left corner of the checkerboard. Let $N(2m, n)$ denote the number of ways the pupils in the black squares can move to their neighboring white seats in a one-to-one manner. In a 4×3 array of desks, one way the pupils in the black squares can move to neighboring white seats is pictured below.



By symmetry, the number of ways the pupils in the white squares can move to their neighboring black seats in a one-to-one manner is $N(2m, n)$. An example of this movement in a 4×3 array of desks is the following.



If we combine the two movements above, we obtain a 4×3 seating rearrangement.



In the general $2m \times n$ case, any combination of a movement of the pupils in the black squares to their neighboring white squares and a movement of the pupils in the white squares to their neighboring black squares (in a one-to-one fashion) results in a unique seating rearrangement. And any seating rearrangement can be decomposed into movement of the pupils in black squares to their neighboring white squares and movement of pupils in white squares into neighboring black squares. Therefore,

$$S(2m, n) = N(2m, n)^2.$$

Hence, the number of seating rearrangements for $2m$ rows with n desks per row is a perfect square!

3. Permanents. A permanent of a square matrix is defined in the same way as the determinant of a square matrix, except that the sign of the permutation is omitted from each term. For a detailed discussion of permanents and their properties, see [5]. For the $2m \times n$ case, number the $2mn$ squares of the checkerboard array in the following manner. The squares in the first row are numbered $1, 2, \dots, n$. The squares in the second row are numbered $n+1, n+2, \dots, 2n$. This process is continued until the squares in the $2m$ th row are numbered $2mn - n + 1, 2mn - n + 2, \dots, 2mn$. Below is an example for the 4×3 case.

1	2	3
4	5	6
7	8	9
10	11	12

Next, form the $2mn \times 2mn$ matrix, L , whose first row consists of 0's and 1's, where a 1 is placed in the column number of the neighbors of the first square. Since the neighbors of the first square are the 2nd and $n + 1$ st squares, row one has 1's in column 2 and column $n + 1$ and 0's everywhere else. Similarly, row 2 has 1's in column 1, 3, and column $n + 2$ and 0's everywhere else. Continue this process for all $2mn$ squares. The resulting matrix L has the form of a partitioned $2m \times 2m$ matrix. That is

$$L = \begin{pmatrix} B & I & 0 & 0 & \dots & 0 & 0 \\ I & B & I & 0 & \dots & 0 & 0 \\ 0 & I & B & I & \dots & 0 & 0 \\ 0 & 0 & I & B & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & B & I \\ 0 & 0 & 0 & 0 & \dots & I & B \end{pmatrix}.$$

Here, B and I are the $n \times n$ matrices where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

By the way the matrices are constructed,

$$S(2m, n) = \text{per } L.$$

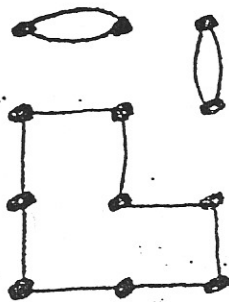
4. **Determinants.** Our next step is to modify L into a matrix \bar{L} such that

$$\text{per } L = \det \bar{L}.$$

That is, we want to replace $l_{r,s}$ by $\bar{l}_{r,s}$ so that

$$\sum_P l_{1,P(1)} \cdots l_{2mn,P(2mn)} = \sum_P (-1)^P \bar{l}_{1,P(1)} \cdots \bar{l}_{2mn,P(2mn)},$$

where P denotes a permutation of $2mn$ objects and $(-1)^P$ denotes the parity of the permutation P . To do this, we need some graph theory. In particular, a seating rearrangement will be represented by a graph. The desks will be the vertices of the graph and the movement in the seating rearrangement will be the edges of the graph. For example, the seating rearrangement in section 2 would be represented by the following graph.

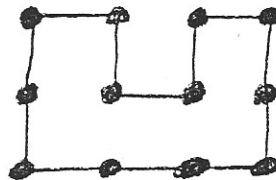


The arrows are not depicted since we will see that they will not be necessary.

Next, we introduce the concept of a single loop. A single loop is a simple closed lattice point path in the coordinate plane (that is, a simple path whose vertices are lattice points) such that each edge of the loop is horizontal or vertical. Some examples of single loops are

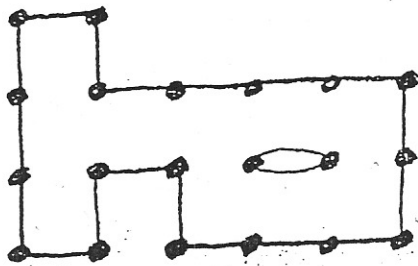


and

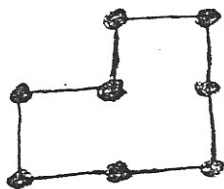


It should be noted that each single loop has an even number of vertices. Now suppose we have a $2m \times n$ seating rearrangement. The graph representing this seating rearrangement consists of a number of single loops. Since each single loop has an even number of vertices, each single loop contributes -1 to the parity of the seating rearrangement permutation. Hence, the parity of the permutation corresponding to a seating rearrangement is the number of single loops in the seating rearrangement.

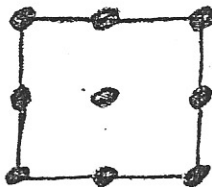
5. Graph Theory. Let R denote a collection of single loops. Also, let $h(R)$ denote the number of horizontal links in R and $v(R)$ denote the number of vertical links in R . In addition, let $L(R)$ denote the number of single loops in R . For the R representing the 4×3 seating rearrangement above, we have $L(R) = 3$, $h(R) = 6$, and $v(R) = 6$. The collection of single loops, R , given below has $L(R) = 2$, $h(R) = 12$, and $v(R) = 8$.



We also need another definition. For a single loop R , let $I(R)$ denote the number of interior points in R . For example

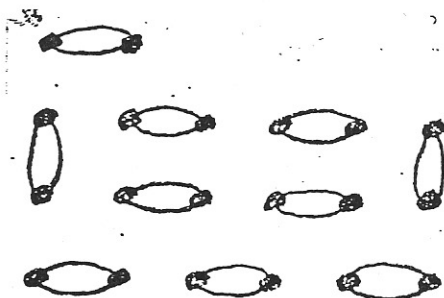


has no interior points and



has one interior point.

We next wish to transform a collection of single loops, R , into a collection of single loops \bar{R} . In this transformation, each single loop in R is traversed and in the process, the odd links are doubled and the even links are dropped to obtain \bar{R} . Where we start is immaterial to the results. For example, our last collection of single loops R has the following as an associated \bar{R} .

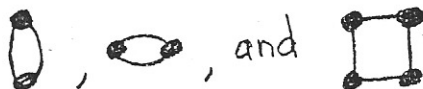


Now we need the following result from graph theory.

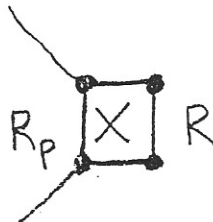
Lemma. Let R be a single loop and \bar{R} be a transformation of R . Then

$$\frac{1}{2}v(R) + \frac{1}{2}h(\bar{R}) + I(R) \equiv 1 \pmod{2}.$$

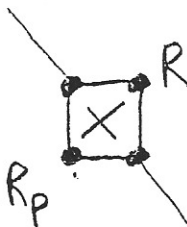
Proof. The proof of this lemma is by induction on the area enclosed by a single loop R . For the zero area loops and the single loop of area one



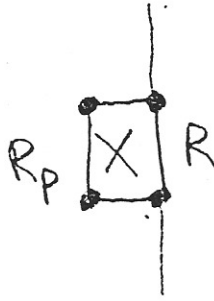
the theorem is true. Now suppose that the theorem is true for any single loop covering k unit squares. Let R be a single loop which covers $k + 1$ unit squares. The single loop R can be thought of as being built from a previous single loop R_p covering k unit squares and adding a unit square having one, two, or three sides in common with R_p . In the case of adding a unit square having one side in common with R_p we have the diagram



Here, $I(R) = I(R_p)$. In the case of adding a unit square having two sides in common with R_p we have the picture



Here, $I(R) = I(R_p) + 1$. Finally in the case of adding a unit square having three sides in common with R_p we have the diagram



Here, $I(R) = I(R_p) + 2$. In each case, we consider $v(R)$, $h(\bar{R})$, $v(R_p)$, and $h(\bar{R}_p)$. Using the induction hypothesis in each case (R_p satisfies the congruence), we can show that R satisfies the congruence. Therefore, by mathematical induction, the lemma is true.

Now suppose we have a $2m \times n$ seating rearrangement. Let R denote the collection of single loops in this $2m \times n$ seating rearrangement. Now the number of interior points in each of these single loops must be even, since the interior points graph of each single loop of a seating rearrangement must be a seating rearrangement for these interior points. Thus, $L(R)$ is congruent to

$$\frac{1}{2}v(R) + \frac{1}{2}h(\bar{R})$$

modulo 2. Next, since R is a collection of single loops corresponding to a seating rearrangement with $2m$ rows,

$$\frac{1}{2}h(\bar{R})$$

is even. Hence for R , the collection of single loops in a $2m \times n$ seating rearrangement,

$$(-1)^P = (-1)^{L(R)} = (-1)^{\frac{1}{2}v(R)} = i^{v(R)}.$$

Here, i is the square root of -1 . Therefore, if

$$\bar{l}_{r,s} = \begin{cases} i, & r \text{ and } s \text{ are vertical neighbors;} \\ 1, & r \text{ and } s \text{ are horizontal neighbors.} \end{cases}$$

then

$$\text{per } L = \det \bar{L}.$$

Thus, we have the $2m \times 2m$ partitioned matrix

$$\bar{L} = \begin{pmatrix} B & iI & 0 & 0 & \dots & 0 & 0 \\ iI & B & iI & 0 & \dots & 0 & 0 \\ 0 & iI & B & iI & \dots & 0 & 0 \\ 0 & 0 & iI & B & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & B & iI \\ 0 & 0 & 0 & 0 & \dots & iI & B \end{pmatrix}.$$

Here, B and I are as before.

6. The Determinant of \bar{L} . Since all the submatrices of \bar{L} commute, they can be treated as ordinary numbers. Thus, if the eigenvalues of the $2m \times 2m$ matrix

$$B' = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

are $\lambda'_1, \lambda'_2, \dots, \lambda'_{2m}$, then by a similarity transformation,

$$\bar{L} = BI' + iB',$$

(where I' is the $2m \times 2m$ identity matrix) can be put in the form

$$\begin{pmatrix} B + i\lambda'_1 I & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & B + i\lambda'_2 I & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & B + i\lambda'_3 I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & B + i\lambda'_4 I & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & B + i\lambda'_{2m-1} I & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & B + i\lambda'_{2m} I \end{pmatrix}.$$

Hence,

$$\det \bar{L} = \prod_{t=1}^{2m} \det(B + i\lambda'_t I).$$

But if the eigenvalues of B are $\lambda_1, \dots, \lambda_n$, then

$$\det(B + i\lambda'_t I) = \prod_{s=1}^n (\lambda_s + i\lambda'_t).$$

Therefore,

$$S(2m, n) = \det \bar{L} = \prod_{t=1}^{2m} \prod_{s=1}^n (\lambda_s + i\lambda'_t).$$

7. Eigenvalues of B . To find the eigenvalues of B , we let

$$D_p = \det \begin{pmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -\lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -\lambda & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -\lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -\lambda \end{pmatrix},$$

where the matrix is of size $p \times p$. Expanding this determinant by its first row we have

$$D_p = -\lambda D_{p-1} - D_{p-2},$$

with initial conditions

$$D_1 = -\lambda \quad \text{and} \quad D_0 = 1.$$

The difference equation for D_p has characteristic equation

$$x^2 + \lambda x + 1 = 0,$$

and if we set $\lambda = -2 \cos \theta$, the roots are

$$x_1 = e^{i\theta} \quad \text{and} \quad x_2 = e^{-i\theta}.$$

Therefore,

$$D_p = ax_1^p + bx_2^p,$$

and using the boundary conditions

$$D_p = \frac{1}{2i \sin \theta} ((e^{i\theta})^{p+1} - (e^{-i\theta})^{p+1}).$$

Thus, the eigenvalues of B are given by

$$\lambda_s = -2 \cos \frac{s\pi}{n+1}, \quad s = 1, \dots, n.$$

8. Theorem and Corollaries.

Theorem.

$$S(2m, n) = 2^{2mn} \prod_{t=1}^m \prod_{s=1}^n \left(\cos^2 \frac{s\pi}{n+1} + \cos^2 \frac{t\pi}{2m+1} \right).$$

Proof. From the discussion in the last two sections,

$$S(2m, n) = 2^{2mn} \prod_{t=1}^{2m} \prod_{s=1}^n \left(\cos \frac{s\pi}{n+1} + i \cos \frac{t\pi}{2m+1} \right).$$

Reducing this expression by multiplying the conjugate terms, we obtain the desired formula.

Next, we give some numerical results.

Corollary 1.

$$\begin{aligned} S(4, 3) &= 121, \\ S(4, 4) &= 1296, \\ S(4, 5) &= 9025, \\ S(6, 6) &= 45265984, \\ S(6, 7) &= 994077841, \\ S(8, 7) &= 1671065533809, \\ S(8, 8) &= 168709341081856, \\ S(10, 10) &= 66865709036047973991424. \end{aligned}$$

The final result follows from [2] and the above theorem.

Corollary 2. Let n be a positive integer. Then

$$F_n^2 = \prod_{j=1}^{n-1} \left(3 + 2 \cos \frac{2j\pi}{n} \right),$$

where F_n denotes the n th Fibonacci number.

References

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