A GENERALIZATION OF A RESULT BY TROLLOPE ON DIGITAL SUMS

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1. Introduction. Let $s_b(i)$ denote the sum of the digits in the base b representation of the nonnegative integer i. Bush [1] showed that

$$\sum_{n<x} s_b(n) \sim \frac{b-1}{2} x \log_b x.$$

Here, $\log_b x$ denotes the base b logarithm of x. Mirsky [8], and later Cheo and Yien [2], proved that

$$\sum_{n<x} s_b(n) = \frac{b-1}{2} x \log_b x + O(x).$$

Trollope [9] discovered the following result. Let $g(x)$ be periodic of period one and defined on $[0,1]$ by

$$g(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(1-x), & \frac{1}{2} < x \le 1, \end{cases}$$

and let

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} g(2^i x).$$

Now, if $n = 2^m(1+x)$, $0 \le x < 1$, then

$$\sum_{i<n} s_2(i) = \frac{1}{2} n \log_2 n - E_2(n),$$

where

$$E_2(n) = 2^{m-1}\Big(2f(x) + (1+x)\log_2(1+x) - 2x\Big).$$

Delange [4] and later Grabner, Kirschenhofer, Prodinger, and Tichy [5] proved an extension of this result. That is,

$$\sum_{n<N} s_b(n) = \frac{b-1}{2} N \log_b N + N\delta(\log_b N),$$

where $\delta(u)$ is a fractal function. Delange's approach was based on a combinatorial decomposition of binary representations of integers, followed by a computation of the Fourier coefficients of the fractal function. Grabner's method used an application of the Mellin transform. Here, we will use Trollope's method to prove a generalization of Trollope's result. We will conclude the paper with some open questions.

2. Notation and Basic Results. To make some of the results easier to state, we will use the notation

$$S(n) = \sum_{i<n} s_b(i).$$

Our first result is a statement about the sum of the base b digital sums in the sequence of positive integers up to a digit times a power of b. The proof of this formula follows from a straightforward counting argument and will be omitted.

Lemma 1. Let d be a nonzero base b digit and m a nonnegative integer. Then

$$S(db^m) = \frac{(b-1)d}{2} m b^m + \frac{(d-1)d}{2} b^m.$$

Next, let n be a positive integer with base b representation

$$n = \sum_{k=0}^{m} d_k b^k = d_m b^m + n_{m-1}.$$

Now, we make the important observation that

$$S(d_m b^m + n_{m-1}) = S(d_m b^m) + d_m n_{m-1} + S(n_{m-1}).$$

Hence, using mathematical induction on the number of digits in n, the above equation, and Lemma 1, we have the following more general result.

Lemma 2. Let n be a positive integer with base b representation

$$n = \sum_{k=0}^{m} d_k b^k$$

and define

$$n_i = \sum_{k=0}^{i} d_k b^k, \quad \text{for} \quad 0 \le i \le m-1; \quad n_{-1} = 0.$$

Then

$$S(n) = \frac{b-1}{2}\sum_{k=0}^{m} d_k k b^k + \frac{1}{2}\sum_{k=0}^{m}(d_k - 1)d_k b^k + \sum_{k=0}^{m} d_k n_{k-1}.$$

3. The Remainder Term. The next step in analyzing $S(n)$ is to study it from a different perspective. We need the following definition.

<u>Definition</u>. Let n be a positive integer with base b representation

$$n = \sum_{k=0}^{m} d_k b^k$$

and again define

$$n_i = \sum_{k=0}^{i} d_k b^k, \quad \text{for } 0 \le i \le m-1; \quad n_{-1} = 0.$$

Let

$$R(n) = \frac{b-1}{2}mn - \frac{b-1}{2}\sum_{k=0}^{m} d_k k b^k - \frac{1}{2}\sum_{k=0}^{m-1}(d_k - 1)d_k b^k - \sum_{k=0}^{m-1} d_k n_{k-1}.$$

The next lemma will state some properties of R, which will be extremely useful throughout the rest of the paper.

<u>Lemma 3</u>.

(a) For any positive integer n, $R(bn) = bR(n)$.

(b) Let n be a positive integer with base b representation

$$n = \sum_{k=0}^{m} d_k b^k.$$

Then

$$R(n+1) - R(n) = \frac{b-1}{2}m + d_m - s_b(n).$$

(c) Let m be a nonnegative integer, d a digit, and p an integer such that $0 \le p < b^{m+1} - b^m$. Then

$$R(b^{m+1} + bp + d) - dR(b^m + p + 1) - (b-d)R(b^m + p) = \frac{(b-d)d}{2}.$$

Proof. The proof of (a) involves a fairly easy, but tedious, derivation using the definition of $R(n)$. In passing, we note that using Lemma 3(a) and the fact that

$$S(n) = \frac{b-1}{2}mn + \frac{(d_m-1)d_m}{2}b^m + d_m n_{m-1} - R(n),$$

it is immediate that

$$S(bn) = bS(n) + \frac{(b-1)b}{2}n$$

for all $n \geq 1$.

The proof of (b) follows from scrutinizing three cases. The first case is when n and $n+1$ have a different number of digits. Thus, $n = b^{m+1} - 1$. The second case is when n and $n+1$ have the same number of digits but have a different first digit. Thus $n = db^m - 1$ for $d = 2, 3, \ldots, b-1$. The third case is the rest of the story, i.e., when n and $n+1$ have the same number of digits and the same first digit. In every case, we have that

$$R(n+1) - R(n) = \frac{b-1}{2}m + d_m - s_b(n).$$

The proof of (c) is a little more involved. Using Lemma 3(b) twice, Lemma 3(a) once, and the assumption that the base b representation of $b^m + p$ is

$$b^m + p = \sum_{k=0}^{m} d_k b^k,$$

we have the following sequence of equalities.

$$\begin{aligned}
&R(b^{m+1}+bp+d)-dR(b^m+p+1)-(b-d)R(b^m+p)\\
&=R(b^{m+1}+bp+d)-d\big(R(b^m+p+1)-R(b^m+p)\big)-bR(b^m+p)\\
&=R(b^{m+1}+bp+d)-R(b^{m+1}+bp)-d\left(\frac{b-1}{2}m+d_m-s_b(b^m+p)\right)\\
&=\sum_{k=0}^{d-1}\big(R(b^{m+1}+bp+k+1)-R(b^{m+1}+bp+k)\big)-\frac{d(b-1)}{2}m\\
&-d_md+ds_b(b^m+p)\\
&=\sum_{k=0}^{d-1}\left(\frac{b-1}{2}(m+1)+d_m-s_b(b^{m+1}+bp+k)\right)-\frac{d(b-1)}{2}m\\
&-d_md+ds_b(b^m+p)\\
&=\frac{d(b-1)}{2}m+\frac{d(b-1)}{2}+dd_m-\sum_{k=0}^{d-1}s_b(b^{m+1}+bp+k)-\frac{d(b-1)}{2}m\\
&-d_md+ds_b(b^m+p)\\
&=\frac{(b-d)d}{2}.
\end{aligned}$$

This completes the proof of (c) and the proof of Lemma 3.

4. Some Functions. Let m be a nonnegative integer and p be an integer such that $0\le p<b^{m+1}-b^m$. Define the function $\phi(x)$ by

$$\phi\left(\frac{p}{b^{m+1}-b^m}\right)=\frac{R(b^m+p)}{b^m}.$$

Note that by Lemma 3(a), $\phi(x)$ is uniquely defined. To see this, suppose x has any other representation, i.e.

$$\frac{p'}{b^{m'+1}-b^{m'}}.$$

Then

$$p'=b^{m'-m}p.$$

Now assume, without loss of generality, that $m'>m$. Then $m'-m$ is a positive integer. Therefore,

$$\frac{R(b^{m'}+p')}{b^{m'}}=\frac{R(b^m+p)}{b^m}.$$

The function $\phi(x)$ is defined only on a subset of $[0,1]$. We now consider the problem of extending this function continuously to $[0,1]$. Here, we solve this problem by considering the limit of a sequence of "polygonal" functions which identify with $\phi(x)$ on the rationals of the form

$$\frac{p}{b^{m+1}-b^m}.$$

These polygonal functions are defined in the following way. Let m be a nonnegative integer. $f_m(x)$ is defined on $[0,1]$ to be the function whose graph is the polygon joining the points

$$\left\{(0,0), \left(\frac{1}{b^{m+1}-b^m}, \phi\left(\frac{1}{b^{m+1}-b^m}\right)\right), \right.$$
$$\left. \cdots, \left(\frac{p}{b^{m+1}-b^m}, \phi\left(\frac{p}{b^{m+1}-b^m}\right)\right), \cdots, (1,0)\right\}.$$

Then, the definition of $\{f_m(x)\}$ is extended to the reals by $f_m(x \pm 1) = f_m(x)$.

From the definition, $f_0 = 0$. In addition, $f_1(x)$ is periodic of period $1/(b-1)$ and on $[0, 1/(b-1)]$ is equal to the piecewise linear function connecting the points

$$\left(\frac{d}{b^2-b}, \frac{(b-d)d}{2b}\right),$$

where d is a nonnegative integer and $0 \leq d \leq b$. For ease of notation, let $g(x) = f_1(x)$. Using the definition of the f's and a special case of Lemma 3(c), it follows that for any nonnegative integer m and all real x,

$$f_{m+1}(x) - f_m(x) = \frac{1}{b^m} g(b^m x).$$

Repeated iterations of this equation yields

$$f_{m+1}(x) = \sum_{i=0}^{m} \frac{1}{b^i} g(b^i x).$$

Since $g(x)$ is bounded, the sequence $\{f_m(x)\}$ converges uniformly for all x. Hence, the limiting function

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{b^i} g(b^i x)$$

is a continuous extension of $\phi(x)$.

5. The Main Result.

Theorem. Let $g(x)$ be periodic of period $1/(b-1)$ and on $[0, 1/(b-1)]$ be equal to the piecewise linear function connecting the points

$$\left(\frac{d}{b^2-b}, \frac{(b-d)d}{2b}\right),$$

where d is a nonnegative integer and $0 \le d \le b$. Let

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{b^i} g(b^i x).$$

Next, let n be a positive integer with base b representation and let

$$n = \sum_{k=0}^{m} d_k b^k = d_m b^m + n_{m-1} = b^m + p = b^m(1 + (b-1)x).$$

Then

$$\sum_{i<n} s_b(i) = \frac{b-1}{2} n \log_b n - E_b(n),$$

where

$$E_b(n) = b^m \biggl(f(x) + \frac{b-1}{2}(1 + (b-1)x) \log_b(1 + (b-1)x) - d_m(1 - d_m + (b-1)x) - \frac{(d_m-1)d_m}{2} \biggr).$$

Proof. Using Lemma 2 and the definition of $R(n)$, it follows that

$$\begin{aligned} S(n) &= \frac{b-1}{2} mn + \frac{(d_m-1)d_m}{2} b^m + d_m n_{m-1} - R(n) \\ &= \frac{b-1}{2} mn + \frac{(d_m-1)d_m}{2} b^m + d_m n_{m-1} - b^m f(x). \end{aligned}$$

Next, since

$$n = b^m(1 + (b-1)x),$$

$$m = \log_b n - \log_b(1 + (b-1)x).$$

Also,

$$n_{m-1} = b^m(1 - d_m + (b-1)x).$$

Substituting for m and n_{m-1} and simplifying, we have

$$S(n) = \frac{b-1}{2} n \log_b n - \frac{b-1}{2} n \log_b (1 + (b-1)x) - b^m f(x)$$
$$+ d_m b^m (1 - d_m + (b-1)x) + \frac{(d_m - 1)d_m}{2} b^m.$$

Finally, substituting for n in the second term the result follows.

6. Questions. Some open questions remain. One problem is to study the function f in the Theorem. For $b = 2$, the function $f/2$ was studied in [7]. Translating those results over to f, when $b = 2$ we have that the maximum value of f is $1/3$. Furthermore, this maximum is attained precisely on the set E consisting of all numbers that can be represented as the infinite quaternary fraction $0.\alpha_1\alpha_2 \ldots \alpha_n \ldots$, where every α_i is either one or two. What can be said about our function f for the general base b? What are the maximum values of f? Where does f attain these maximum values?

Delange's method has been used by Coquet [3] and Kirschenhofer [6] and extended by others to obtain the formula

$$\sum_{n<N} s_b(n)^2 = \left(\frac{b-1}{2}\right)^2 N \log_b^2 N + N \log_b N \eta_1(\log_b N) + N \eta_2(\log_b N),$$

where η_1 and η_2 are continuous nowhere differentiable functions of period 1. Can Trollope's method be used to obtain a similar formula? Also, what can be said about higher moments using Trollope's method?

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