SUPER NIVEN NUMBERS

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1. Introduction. The subject of Niven numbers has been fascinating to some mathematicians working in number theory. Niven numbers were named in honor of Ivan Niven who inspired these investigations in 1977 at the fifth annual Miami University Conference on Number Theory. The definition of a Niven number involves the use of the *digital sum* of a number. For a positive integer n, we will denote the digital sum of n by $s(n)$. For example, the digital sum of 4 is 4 while the digital sum of 111 is 3 since 1+1+1=3. We define a Niven number to be an integer divisible by its digital sum.

Definition 1.1. A positive integer n is a *Niven number* if $s(n) \mid n$.

Examples of Niven numbers are 4, 12, and 111, since they are divisible by the sum of their digits 4, 3, and 3 respectively. The following are some theorems and open questions concerning Niven numbers.

Theorem 1.1. Let $N(x)$ denote the number of Niven numbers not exceeding x. Then, the natural density of the Niven numbers is zero, that is,

$$\lim_{x\to\infty} \frac{N(x)}{x} = 0 \ .$$

This means that there are not many Niven numbers when compared to the set of positive integers. The proof of Theorem 1.1 can be found in [1].

The first non–Niven factorial was shown to be 432! in [2]. In [3], the following necessary condition for a factorial to be Niven was given.

Theorem 1.2. If $s(n!)$ is less than or equal to $18n$ and $s(n!)$ is even, then $n!$ is Niven.

No necessary and sufficient condition in order that $n!$ be Niven has been discovered.

A more difficult question is which powers of 2 are Niven. A power of 2 is Niven if and only if its digital sum is a power of 2. The first few powers of 2 that are Niven, as found in [4], are

$$2^1,\ 2^2,\ 2^3,\ 2^9,\ 2^{36},\ 2^{85},\ 2^{176},\ 2^{194},\ 2^{200},\ 2^{375},\ 2^{1517}\ ,$$

$$2^{1573},\ 2^{3042},\ 2^{5953},\ 2^{6043},\ 2^{6109},\ 2^{12068},\ 2^{12104},\ldots .$$

The powers of 2 that are Niven have not been characterized. It is not even known whether or not there are infinitely many of them.

2. Super Niven Numbers. To extend the concept of Niven number, we define the following.

Definition 2.1. A *subdigital sum* of a positive integer n is a sum of some of the nonzero digits of n. We denote the set of all subdigital sums of n by S_n.

For example,

$$S_{102} = \{1, 2, 1+2=3\}\ .$$

Also,

$$S_{3902} = \{3, 9, 2, 3+9=12, 3+2=5, 9+2=11, 3+9+2=14\}\ .$$

Next, we define the idea of a "Super Niven" number.

Definition 2.2. A positive integer n is a *Super Niven number* if n is divisible by every member of S_n.

The integer 102 is Super Niven since 1, 2, and 3 divide 102. However, 3902 is not Super Niven since 5 does not divide 3902. Also, there are Niven numbers which are not Super Niven. An example of this is the integer 18. We can make two observations. First of all, every integral power of 10 is Super Niven. Thus, there are an infinite number of Super Niven numbers.

If n is Super Niven, then n is Niven. Thus, the set of Super Niven numbers is contained in the set of Niven numbers. Hence, we have the following theorem.

Theorem 2.1. The natural density of the set of Super Niven numbers is 0.

By examining a list of Super Niven numbers, we noticed the following characteristic of Super Niven numbers given in Theorem 2.2.

Theorem 2.2. The only odd Super Niven numbers are 1, 3, 5, 7, and 9.

Proof. Suppose N is an odd Super Niven number consisting of n digits where $n \geq 2$. Thus

$$N = \sum_{i=1}^{n} d_i 10^{i-1} \ ,$$

where each d_i is a digit. In what follows, the notation

$$d_n d_{n-1} \cdots d_1$$

will be used to represent

$$\sum_{i=1}^{n} d_i 10^{i-1} \ .$$

Note that, by hypothesis, d_1 is odd. We need only to investigate the following two cases.

Case 1. d_n is even.

Since by definition of Super Niven numbers,

$$d_n \bigm| N \ ,$$

it follows that N is even.

Case 2. d_n is odd.

Since by hypothesis d_1 is odd, we have that

$$d_n + d_1$$

is even. Again, by definition,

$$d_n + d_1 \bigm| N \ .$$

Hence, in either case, N must be even. Thus, the only odd Super Niven numbers are 1, 3, 5, 7, and 9.

The following theorem is a useful tool for the investigation of Super Niven numbers.

Theorem 2.3. Let n be a Super Niven number with at least k nonzero digits. Then n is divisible by k .

Proof. Suppose n has k nonzero digits,

$$d_1, d_2, d_3, \cdots, d_k \ .$$

Consider the set

$$D = \{d_1, d_1 + d_2, d_1 + d_2 + d_3, \cdots, d_1 + d_2 + \cdots + d_k\} \ .$$

Note that $D \subseteq S_n$. Let us divide every member of D by k and examine the remainders.

$$\begin{aligned} & d_1 \equiv r_1 \pmod{k} \ , \\ & d_1 + d_2 \equiv r_2 \pmod{k} \ , \\ & \qquad \vdots \\ & d_1 + d_2 + \cdots + d_k \equiv r_k \pmod{k} \ . \end{aligned}$$

There are two cases.

Case 1. All the remainders are distinct.

There are k distinct remainders. Then, one of the remainders must be zero. That is,

$$d_1 + d_2 + d_3 + \cdots + d_i \equiv 0 \pmod{k}$$

for some $1 \leq i \leq k$. Thus,

$$k \mid d_1 + d_2 + \cdots + d_i \ .$$

But by the definition of Super Niven numbers,

$$d_1 + d_2 \cdots + d_i \mid n \ .$$

Therefore,

$$k \mid n \ .$$

Case 2. At least two remainders are the same.

Suppose two of the remainders are equal. That is,

(1) $$d_1 + d_2 + \cdots + d_i \equiv r \pmod{k}$$

and

$$(2) \qquad d_1 + d_2 + \cdots + d_j \equiv r \pmod{k} ,$$

where $i > j$. Subtracting (2) from (1),

$$(d_1 + d_2 + \cdots + d_i) - (d_1 + d_2 + \cdots + d_j) \equiv r - r \pmod{k} .$$

Thus,

$$d_{j+1} + d_{j+2} + \cdots + d_i \equiv 0 \pmod{k} .$$

Therefore,

$$k \mid d_{j+1} + d_{j+2} + \cdots + d_i .$$

But again, by the definition of Super Niven numbers,

$$d_{j+1} + d_{j+2} + \cdots + d_i \mid n .$$

Hence,

$$k \mid n .$$

Next, we investigate some consequences of Theorem 2.3.

Corollary 2.1. Let $n \geq 10^4$ be a Super Niven number. Then n has a zero digit.

Proof. By contradiction. Suppose $n \geq 10^4$ is Super Niven and n has all nonzero digits. Then n has at least 5 nonzero digits. (Also, n has at least 2 nonzero digits.) By Theorem 2.3, n is divisible by 5 (and 2). Thus n is divisible by 10 so the units digit of n is 0. This is a contradiction.

As a direct result of Corollary 2.1 we have the following corollary.

Corollary 2.2. The only Super Niven numbers with all nonzero digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 24, 36, and 48.

Proof. By Corollary 2.1, we know that if n is Super Niven and $n \geq 10^4$, then n has a zero digit. Thus no Super Niven number greater than or equal to 10^4 can have all nonzero digits. It is straightforward to computer–generate the list of all one hundred forty–two Super Niven numbers from 1 to 10000 and to note that all except for the thirteen listed above have at least one zero digit.

The following corollaries are direct consequences of Theorem 2.3 and are given without proof.

Corollary 2.3. No Super Niven number has a digital sum of 11.

Corollary 2.4. The only possible Super Niven numbers with a digital sum of 13 have exactly 2 nonzero digits, 8 and 5.

Corollary 2.5. Let n be a Super Niven number and $s(n) \geq 19$. Then $s(n)$ is divisible by 3.

3. When is 2^n Super Niven? A discussion of when 2^n is Super Niven will use a preliminary result by Schinzel [5].

Lemma 3.1. If g is an even positive integer not divisible by 10, then the sum of the decimal digits of g^n increases to infinity with n.

The method he used to prove this result will be used as a model for the proof of Theorem 3.1 below. For completeness, we include Schinzel's proof of the above lemma.

Proof. Let us define an infinite sequence of integers a_i $(i = 0, 1, 2, \cdots)$ as follows: put $a_0 = 0$, and for $k = 0, 1, 2, \cdots$ let a_{k+1} denote the smallest positive integer such that $2^{a_{k+1}} \geq 10^{a_k}$ (thus, we shall have $a_1 = 1$, $a_2 = 4$, $a_3 = 14$, and so forth). Clearly, $a_1 < a_2 < a_3 < \cdots$.

We shall prove that if for some positive integer k we have $n \geq a_k$, then the sum of digits of g^n is greater than or equal to k.

Let c_j denote the digit of the decimal expansion of g^n standing at 10^j. Since g is even, we have $2^n \mid g^n$, and since $n \geq a_k$, we have, for $i = 1, 2, \cdots, k-1$, the relation $2^{a_i} \mid g^n$. Moreover, since $2^{a_i} | 10^{a_i}$, we have

$$2^{a_i} \mid c_{a_i - 1} 10^{a_i - 1} + \cdots + c_0 \ .$$

If for $a_{i-1} \le j < a_i$ all digits c_j were equal zero, we would have

$$2^{a_i} \mid c_{a_{i-1}-1} 10^{a_{i-1}-1} + \cdots + c_0 \ ,$$

and, in view of $c_0 \neq 0$, also

$$2^{a_i} \le c_{a_{i-1}-1} 10^{a_{i-1}-1} + \cdots + c_0 < 10^{a_i - 1} \ .$$

This implies $2^{a_i} < 10^{a_i - 1}$, contrary to the definition of a_i. Thus, at least one of the digits c_j, where $a_{i-1} \le j < a_i$, is different from zero. Since this is true for $i = 1, 2, \cdots, k$, at least k digits of g^n are different from zero. For sufficiently large n (for $n \ge a_k$), the sum of decimal digits of g^n is not smaller than an arbitrarily given number k. This shows that the sum of decimal digits of g^n increases to infinity together with n, which was to be proved.

Using Lemma 3.1, we give a theorem which will lead to a corollary concerning the possibility that 2^n is Super Niven.

Theorem 3.1. Let n be a positive integer. Then

$$s(2^n) \ge \log_4 n - 1 \ .$$

Proof. In the proof of Lemma 3.1, Schinzel defines the sequence $a_0 = 0$ and a_{k+1} denotes the smallest positive integer such that

$$2^{a_{k+1}} > 10^{a_k} \ ,$$

where $k \ge 0$. Thus

$$a_{k+1} > \log_2 10^{a_k} = a_k \cdot \log_2 10 = \frac{a_k}{\log 2} \ ,$$

where log denotes the base 10 logarithm. Therefore, $a_1 = 1$ and

$$a_{k+1} = \left\lceil \frac{a_k}{\log 2} \right\rceil \ ,$$

for $k \ge 1$. Here, $\lceil \cdot \rceil$ denotes the ceiling function. But

$$a_{k+1} = \left\lceil \frac{a_k}{\log 2} \right\rceil \le 4 a_k \ ,$$

so

$$a_{k+1} \leq 4a_k \leq \cdots \leq 4^k a_1 = 4^k < 4^{k+1} \ .$$

Next, in the proof of Lemma 3.1, Schinzel proves that if $n \geq a_k$, then $s(2^n) \geq k$. Therefore, if $n \geq 4^k$, then $s(2^n) \geq k$. But

$$n = 4^{\log_4 n} \geq 4^{\lfloor \log_4 n \rfloor} \ ,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Thus,

$$s(2^n) \geq \lfloor \log_4 n \rfloor \geq \log_4 n - 1 \ .$$

Corollary 3.1. If $n \geq 2^{40}$, then 2^n is not Super Niven.

Proof. By contradiction. Suppose $n \geq 2^{40}$ and 2^n is Super Niven. Since $n \geq 2^{40}$,

$$\log_4 n - 1 \geq 19 \ .$$

By Theorem 3.1, $s(2^n) \geq 19$. Thus 2^n has at least 3 nonzero digits. Since 2^n is Super Niven, by Theorem 2.3, 2^n is divisible by 3, a contradiction.

4. Conclusion. Finally, we present the following conjectures, open questions, and topics for further research. Based on a large list of numerical evidence, we believe that

Conjecture 4.1. If $n \geq 9$ is an integer, then $n!$ is not Super Niven.

We can, however, give the following heuristic argument that this conjecture is indeed true. By [6], the number of digits in $n!$ is

$$n \log n + O(n) \ .$$

(A good discussion of the "Big–Oh" notation can be found in [7].) In [8] it is shown that the number of terminating zeros of $n!$ is precisely

$$\sum_{i \geq 1} \left[\frac{n}{5^i} \right] = O(n) \ .$$

Therefore, the number of possible nonzero digits in $n!$ is

$$n \log n + O(n) \ .$$

Assuming that these digits are uniformly and randomly distributed among the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, it seems plausible to assume that the number of each digit in $n!$ is

$$\frac{1}{10} n \log n + O(n) \ .$$

Furthermore, from all these nonzero digits, we should be able to construct every subdigital sum of $n!$ from 1 to $\frac{9}{2} n \log n$. But from a generalization of Bertrand's Postulate, there is a prime p between n and $\frac{3}{2}n$ for $n \geq 9$. This would mean that $n!$ would not be Super Niven.

By stating the contrapositive to Theorem 2.3, we have a way to tell if a number is not Super Niven.

Theorem 4.1. Suppose n has at least k nonzero digits and n is not divisible by k. Then n is not Super Niven.

For example, 142 is not Super Niven since 142 has at least 3 nonzero digits and 3 does not divide 142.

Theorem 4.1 gives a subject for further investigation , the idea of a pseudo–Super Niven number.

Definition 4.1. A positive integer n is *pseudo–Super Niven* if n is not Super Niven and $k \bigm| n$ for all k less than or equal to the number of nonzero digits in n.

For example, 114 is a pseudo–Super Niven number since 1, 2, and 3 divide 114 and 114 is not Super Niven (since 1+4=5 does not divide 114). A larger example of a pseudo–Super Niven number is 163380. Note that 1, 2, 3, 4, and 5 divide 163380 but 163380 is not Super Niven since 3+8=11 does not divide 163380.

It would be interesting to know "how many" pseudo–Super Niven numbers exist. We conclude with the following open questions.

Question 4.1. Are there an infinite number of pseudo–Super Niven numbers?

We showed that eventually, the powers of 2 are not Super Niven. We would like to find all powers of 2 which are Super Niven. This study may require a supercomputer.

Question 4.2. What powers of 2 are Super Niven?

Finally, we should mention that the definitions and results given here can be extended to other bases. This investigation is left for future study.

References

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