ABSTRACT

Tau Numbers, Natural Density, and Hardy and Wright's Theorem 437

by

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An element of the set

$$T = \{n : \tau(n) \text{ is a factor of } n\}$$

is called a Tau number, where $\tau(n)$ denotes the number of divisors of the integer n. We determine the natural density of this set by use of Hardy and Wright's Theorem 437 (4th ed.).

1. **Introduction.** Hardy and Wright's, *An Introduction to the Theory of Numbers* [1] is one of the classic references for number theorists. If one has a number theoretic question, chances are that it is either answered or will be a corollary to a result contained in this book. This note relates how the latter possibility can occur.

Recently, a colleague of ours (Alvin Tinsley) was searching for suitable questions that concerned the divisors of an integer. One such question was, "Find an integer n such that $\tau(n)$ divides n" where $\tau(n)$ denotes the number of divisors of n. He, of course, found many such integers that met this condition. However, for his own curiosity, he investigated whether or not such integers could be characterized. Such a characterization is still an open problem. However, since we are often interested in the concept of natural density, we also investigated the existence of the natural density of this set of integers. In what follows, we outline the steps that show that the natural density of this set is 0.

2. **Notation and Terminology.** For a set, A, of integers, $A(x)$ denotes the number of members of A not exceeding x. If

$$\lim_{x\to\infty} \frac{A(x)}{x}$$

exists, we call this limit the natural density of A, and is symbolized by $\delta(A)$. Here we consider the set of positive integers

$$T = \{n : \tau(n) \text{ is a factor of } n\}$$

and determine $\delta(T)$. In what follows, an element of T will be called a Tau number. Thus, we will determine $\delta(T)$ in order to show that the natural density of T is zero. Any undefined notation or terminology can be found in the basic references [1] and [2].

3. **The Natural Density of $K \cap T$.** Let k be a fixed positive integer, and K be the collection of k-free integers. Then $n \in K \cap T$ implies that n is both k-free and $\tau(n)$ is a factor of n. If the prime factorization of n is

$$\prod_{i=1}^{t} p_i^{n_i} ,$$

then

$$\tau(n) = \prod_{i=1}^{t} (n_i + 1) .$$

Let

$$\Omega(n) = \sum_{i=1}^{t} n_i ,$$

and note that since n is k-free, $\tau(n)$ can also be written in the form

$$\tau(n) = 2^{m_1} 3^{m_2} 4^{m_3} \cdots k^{m_{k-1}}$$

where $m_j = \#\{n_i : n_i = j\}$. Also since $n \in T$, $m_j \leq k$. It follows that

$$\Omega(n) = \sum_{j=1}^{k-1} j \cdot m_j \leq \sum_{j=1}^{k-1} k^2 \leq k^3 .$$

This gives an upper bound for $\Omega(n)$ which, though seemingly too large, will be sufficient for our purposes.

Stating this result as a lemma, we have

Lemma 1. Let k be a positive integer, and K be the collection of k-free integers. Then $\Omega(n) \leq k^3$ for each $n \in K \cap T$.

4. **A Corollary to Hardy and Wright's Theorem 437.** In [1; page 368] , the following theorem is presented:

Theorem 2. (Theorem 437) Let $T_t = \{n : \Omega(n) = t\}$. Then,

$$T_t(x) \sim \frac{x\,(\log\log x)^{t-1}}{(t-1)!\,\log x} .$$

Since by Lemma 1,

$$K \cap T \subseteq \bigcup_{t=1}^{k^3} T_t \ ,$$

it follows that

$$(K \cap T)(x) \leq \sum_{t=1}^{k^3} T_t(x) \ .$$

Hence by Theorem 2,

$$\begin{aligned}
\lim_{x\to\infty} \frac{(K \cap T)(x)}{x} &\leq \lim_{x\to\infty} \sum_{t=1}^{k^3} \frac{T_t(x)}{x} \\
&= \sum_{t=1}^{k^3} \lim_{x\to\infty} \frac{(\log\log x)^{t-1}}{(t-1)!\ \log x} \\
&= 0 \ .
\end{aligned}$$

Thus, $\delta(K \cap T) = 0$, and we have, as a corollary to Hardy and Wright's Theorem 437, that the natural density of the intersection of the set of k-free integers and the Tau numbers is 0.

This result is the key to finding $\delta(T)$. By observing that since $T \subseteq K' \cup (K \cap T)$, we have that

$$\frac{T(x)}{x} \leq \frac{K'(x)}{x} + \frac{(K \cap T)(x)}{x} \ .$$

Here, K' is the complement of K with respect to the positive integers. Thus,

$$\frac{T(x)}{x} \leq 1 - \frac{K(x)}{x}$$

and we see that knowing the natural density of the k-free integers will give some information about $\delta(T)$. In fact, knowing the value of $\delta(K)$ will help to determine that the value of $\delta(T)$ is 0.

5. **The Natural Density of the Tau Numbers.** Using the above results, and the well-known fact that

$$\delta(K) = \frac{1}{\zeta(k)} \ ,$$

we have that

$$\lim_{x \to \infty} \frac{T(x)}{x} \leq 1 - \frac{1}{\zeta(k)}$$

for any $k \geq 2$. But since

$$\lim_{k \to \infty} \zeta(k) = 1 \ ,$$

we conclude that $\delta(T) = 0$, and have proven that the natural density of the Tau numbers is 0.

References

[1] Hardy, G.H. and Wright, E.M., *An Introduction to the Theory of Numbers*, 4th ed. Oxford: Clarendon Press (1960).

[2] Apostol, T.M., *Introduction to Analytic Number Theory*, Springer-Verlag (1976).

[3] Knuth, D.E., *The Art of Computer Programming*, Vol. 1, Addison-Wesley (1969).

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