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A COMPACT STRATEGY FOR NUMERICAL TIC-TAC-TOE

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1. Introduction

Tic-tac-toe is a game that almost everyone has played, usually in grade-school during a class while something "more important" was being ignored (math class being the exception). However, anyone who has played this simple game probably realized fairly quickly that the game could not be won by either side if both players were fairly experienced. A complete analysis of tic-tac-toe appears in [1].

Ron Graham is credited with the discovery of a numerical version of tic-tac-toe that is a much more interesting and challenging game to play. The game is played on the usual tic-tac-toe grid, but instead of using X's and O's, the players use the integers 1–9 with the first player (called ODD) playing the odd numbers and the second player (called EVEN) playing the even numbers. A "play" or "move" means that a player places one of his/her remaining digits in an unoccupied cell of the tic-tac-toe grid. Each number may be played only once and the object of the game is to win by completing any row, column, or diagonal to obtain a sum of 15.

The game is more challenging because either player may complete a line containing numbers placed by the opposing player. Also, the game is more interesting since, with an appropriate strategy, one of the players can win. Using a computer analysis of the game, George Markowski proved that ODD can always win. He offers a strategy involving the use of a rather lengthy look-up table [2].

Presented here is a relatively compact non-look-up-table strategy for ODD to win Graham's game.

2. The Magic Square

Most students of arithmetic have at one time or another been exposed to the wonders of magic squares– the $n \times n$ grids of numbers whose rows, columns, and diagonals all have the same sum. A discussion of magic squares appears in [3]. It should be readily apparent that the magic square constructed on a 3×3 grid using the numbers 1–9 and forming the sum 15 (see Figure 1), will be relevant to the discussion here. A quick check will reveal that all possible winning triples in Graham's game are represented in the rows, columns, and diagonals of this magic square. Actually, what is relevant is that a player needs a list of all three-element subsets of $\{1,2,\ldots,9\}$ which sum to 15. It just so happens that all these subsets appear in the "lines" of this magic square, a coincidence. However, the arrangement of the numbers in the square make it a handy reference to explain and demonstrate ODD's Winning Strategy.

8	1	6
3	5	7
4	9	2

Figure 1.

3. Terminology and Strategy

Before presenting the actual strategy, an assumption and some terminology will be discussed.

First of all, the strategy assumes that the players are not stupid. "Not playing stupid" consists of taking a win when possible, blocking an opponent's win on the next move if necessary, and not giving the opponent a win on the next move unless unavoidable.

Some of the terms used in the strategy are defined in Figure 2.

Side, *center*, and *corner* refer to the cells of the tic-tac-toe grid as shown in Figure 2.

A *line* is any row, column, or diagonal of a game.

An *open line* is any line containing one even number and nothing else.

To *force* is to play a second number in a line while holding the number that would complete the line for a win. The next player's move is *forced* and must *block* the win, if possible, by playing a number in the remaining unoccupied cell of that line.

corner	side	corner
side	center	side
corner	side	corner

Figure 2.

An even number's *odd neighbor* is one of the two odd numbers in the row and column of the magic square containing that even number. For example, 1 and 3 are odd neighbors of 8.

To *attack* is to cause the opponent to eventually play a fatal second number in an open line. ODD attacks an open line by playing outside of the line under attack.

The number to be played in an attack can be determined by a quick check of EVEN's remaining numbers and retaining all available odd numbers required to complete the line in a win. An attack may continue for two moves by ODD in some cases.

To avoid confusion, the strategy indicates the game's move numbers with words and the numbers played with digits. To proceed through the strategy, choose the appropriate case, then the earliest alphabetic instruction that is possible (without playing stupid).

ODD's Winning Strategy

Move one – Play 1, 3, 7, or 9 in any side cell.
Move three –

Case I: Move two *is not* a force.

a) Using 9, 7, 3, or 1, force move four to the center (first choice), or to a corner (second choice).
Move five – Attack all open lines containing move four with move two's remaining odd neighbor.

b) Attack all open lines containing move two

Case II: Move two *is* a force.
Block with move two's remaining odd neighbor.
Move five –

a) Attack the open line containing move four with force.
b) Retaining 5 for the win, force move six to play in both an open line containing move two and the open line containing move four.
c) Attack the open line containing move four by playing in the open line containing move two.

8	1	6
3	5	7
4	9	2

The Magic Square

4. Examples

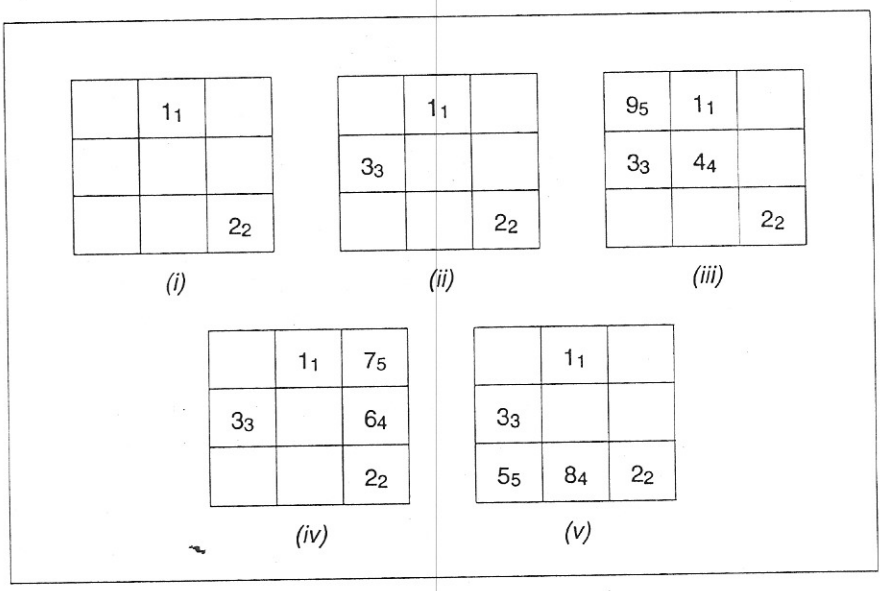
In order to illustrate how ODD's Winning Strategy should be used, several examples are given. In each example, the cells of the tic-tac-toe grid are referred to by letter as indicated in Figure 3. Subscripts indicate moves of the game and asterisks indicate all forced moves.

Consider the game shown in Example 1. After move two (*i*), ODD is not forced so Case I is chosen and line a) is checked. To force move four, ODD must play 9, and according to the strategy, in H (first choice) or in A or C (second choice). Since any of these moves would be "stupid," ODD proceeds to line b). To attack

all open lines containing move 2, ODD determines that 5, 7, and 9 must be retained, leaving 3 to be played in D (*ii*), the only cell outside of all open lines under attack. EVEN's next move is fatal, because ODD has retained all numbers available to complete for a win any line in which move four plays (*iii*), (*iv*), or (*v*).

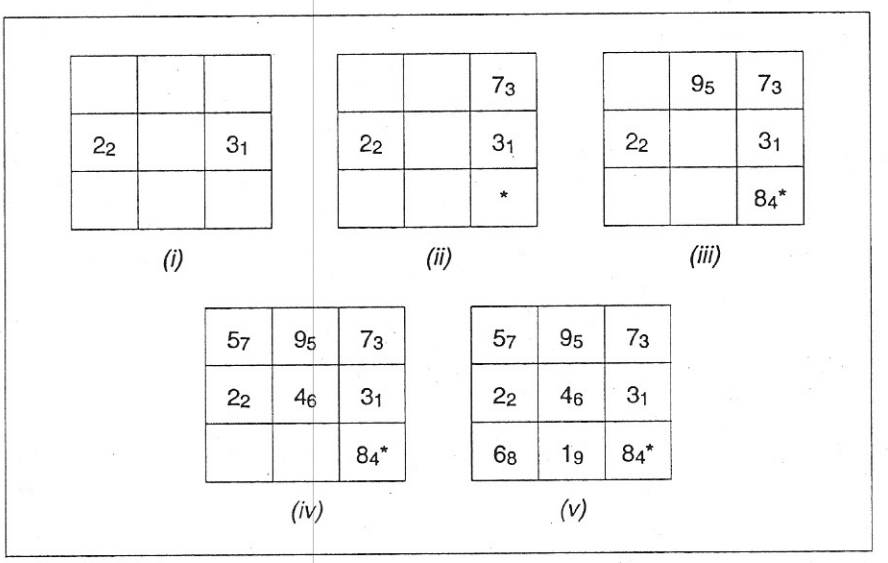
A	B	C
D	E	F
G	H	I

Figure 3.



Example 1.

In Example 2, once again move two is not a force (*i*). Checking Case I a), ODD cannot force move four to the center but can force move four to a corner without playing stupid by playing 7 in C or I (*ii*). Move four can play any of its remaining numbers to block the force, and 8 played in I is considered here (*iii*). Move five proceeds through the strategy by attacking the two open lines containing 8 (move four) with 2's remaining odd neighbor form the magic square, 9. To attack these open lines move five must play in B (*iii*). Notice that 1 was the only number available that would complete one of these lines in a win. To avoid playing stupid, EVEN will extend the game to nine moves by playing 4 in one of the remaining cells (*iv*). Move seven continues to attack the remaining open line containing 8 by playing 5 in A (*iv*). Move eight is fatal because ODD has retained 1 for the win (*v*).

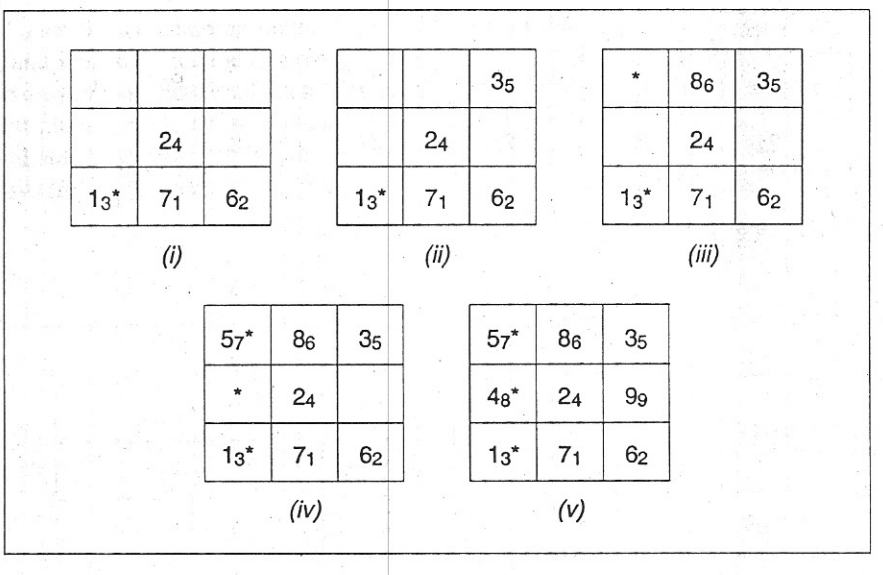


Example 2.

A	B	C
D	E	F
G	H	I

Notice in this example, that the attack continued for two of ODD's moves because ODD did not have *all* of the necessary odd numbers. One of the key ideas in an attack is to hold back the numbers that ODD has *available*. EVEN must eventually play the fatal number that gives ODD the win.

Example 3 illustrates a Case II game. Move two forces and move three blocks with 6's remaining odd neighbor from the magic square, 1(*i*). Move four has several options, one of which is to play the 2 in E (*i*). ODD checks line a) to see if an attack with force is possible. To attack an open line containing 2 (move four), ODD must retain 5 and 9. 3 cannot force with 1 and cannot find a position to force with 7. Therefore, ODD must go to check line b). To force move six to play in both an open line containing 6 (move two) and an open line containing 2, ODD would need to be in a position to force this move into F, but is not. That leaves line c) in the strategy and, retaining 5 and 9, move five attacks the open line containing 2 (move four) by playing 3 in C (*ii*). Once again, EVEN has a couple of options, but any non-stupid move involves forcing move seven (which is really immaterial to ODD at this point). In this example, 8 is played in B (*iii*). While blocking the force, ODD at the same time continues the attack on the open line containing 2 by playing 5 in A (*iv*). The resulting force of



Example 3.

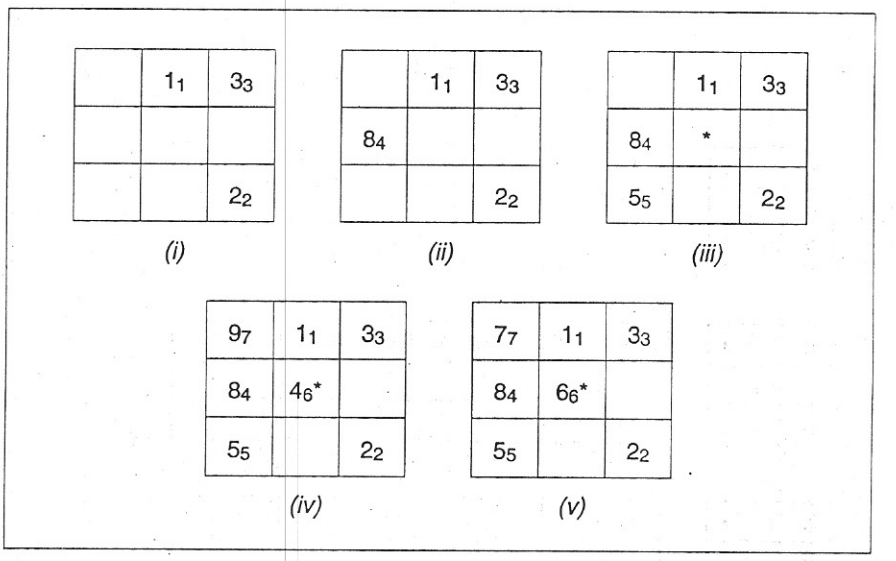
move eight is coincidental and not critical. ODD had retained 9 for the win (v) due to the rules of the attack.

In this example, ODD also continued an attack for more than one move. Remember that once an attack is started, it should continue against the same open line until the winning number can be played.

A	B	C
D	E	F
G	H	I

These examples should give the player designated as ODD an indication of how to proceed through the strategy. And although not necessary to be stated in the strategy, there are a few points that are interesting to note (and helpful to remember). 5 is never played on moves one or three. Also, move three is always played such that move four is never a force. Whenever ODD initiates an attack, it is against the open line(s) containing the last even number played. Also, move two playing a non-forcing number (of the number played in move one) to produce three open lines might be described as "nearly stupid" because ODD can easily win on move five.

As a player gains experience, other patterns will emerge that will make the strategy easier to remember, especially if ODD keeps a magic square on hand and marks off the numbers as they are played.



Example 4.

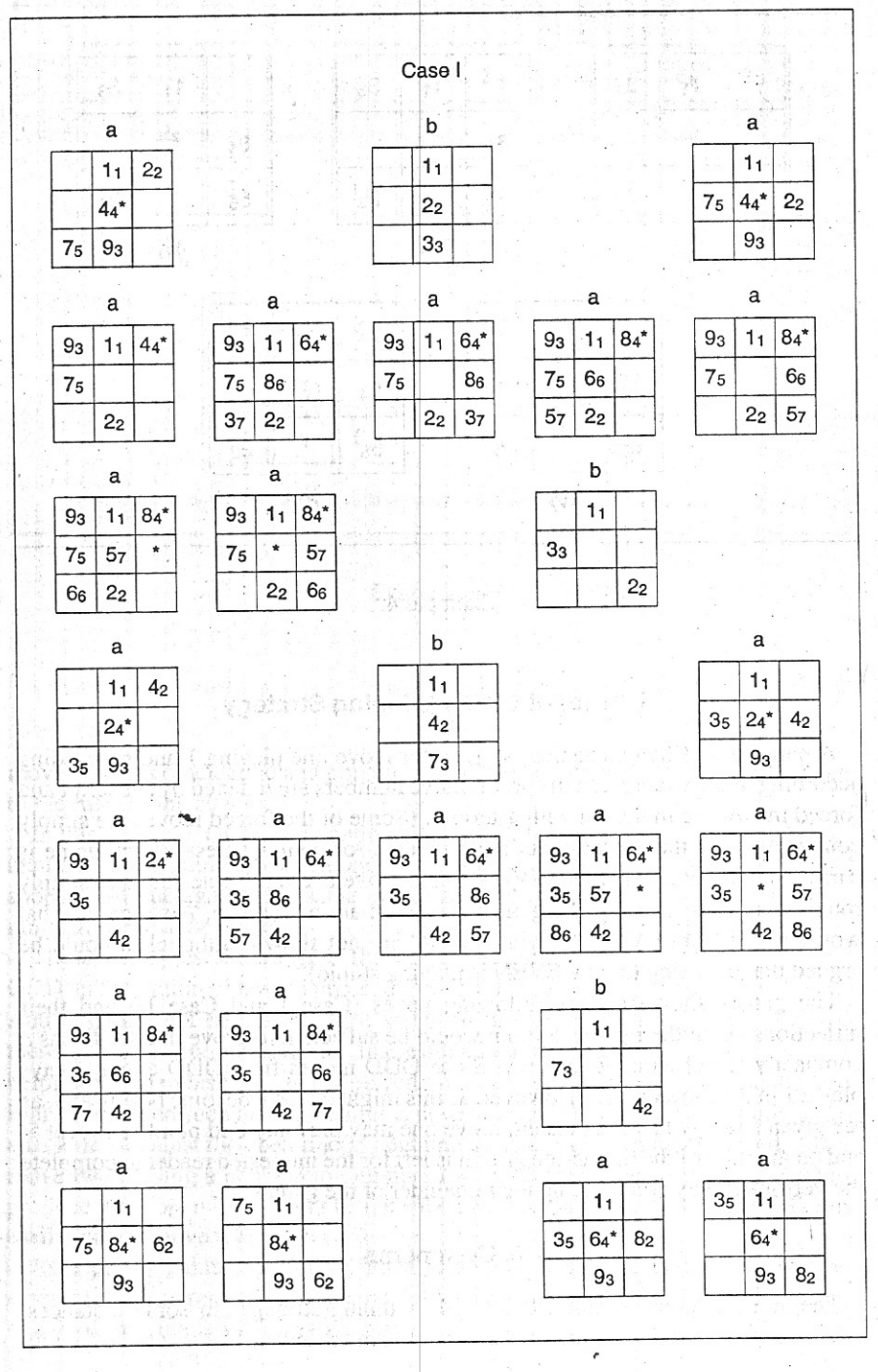
5. Proof of ODD's Winning Strategy

A partial proof by exhaustion is given for move one playing 1 and proceeding according to the strategy. In the proof, move numbers are indexed by subscript and forced moves are indicated with asterisks. (Some of the forced moves are simply coincidences of the strategy, but are indicated for completeness.) Each game is carried out to the point where EVEN's next move is fatal. Games that are simply rotations or reflections of the games presented are not shown. Also, games that would result from EVEN "playing stupid" are not shown (although it could be argued that choosing to play EVEN is playing stupid).

The games shown on the following pages (Case I and Case II) and their reflections about the middle column would be sufficient to prove that ODD has *a* compactly stated winning strategy. Since ODD moves first, ODD could always play a 1 in the top side-cell. However, as this might quickly become boring and, as the given strategy has been stated, move one may play any odd number except 5, and on any side of the tic-tac-toe grid. It is left for the interested reader to complete the verification by constructing the remainder of the games.

6. Comments

The strategy given is not ODD's only winning strategy. In some instances, ODD can make a different move than that indicated and still be assured of a win.


Case I
a
b
a
a
a
a
a
a
b
a
b
a
a
a
a
a
a
b
a
a
a
a

Case II

a

	1_1	
7_5	$4_4{}^*$	2_2
	9_3	

a

9_3	1_1	$8_4{}^*$
7_5		6_6
	2_2	5_7

		$8_4{}^*$

		2_2

a

	1_1	
3_5	$2_4{}^*$	4_2
	9_3	

a

9_3	1_1	$6_4{}^*$
3_5	*	5_7
	4_2	8_6

		$6_4{}^*$
		*

		4_2

a

3_5	1_1	
	$6_4{}^*$	
	9_3	8_2

		8_2

a

$7_3{}^*$	1_1	6_2
2_4	*	
		3_5

a

$7_3{}^*$	1_1	6_2
*	2_4	
3_5		

a

$7_3{}^*$	1_1	6_2
*		2_4
3_5		

a

$7_3{}^*$	1_1	6_2
	3_5	
2_4		*

a

$7_3{}^*$	1_1	6_2
3_5		
*	2_4	

a

$7_3{}^*$	1_1	6_2
3_5		
*		2_4

a

$7_3{}^*$	1_1	6_2
4_4	*	
	5_5	

a

$7_3{}^*$	1_1	6_2
5_5		
*	4_4	

a

$7_3{}^*$	1_1	6_2
8_4	*	
	9_5	

c

$7_3{}^*$	1_1	6_2
	8_4	
2_2	$5_7{}^*$	9_5

c

$7_3{}^*$	1_1	6_2
*	8_4	
$5_7{}^*$	2_6	9_5

c

$7_3{}^*$	1_1	6_2
	8_4	
4_6	$3_7{}^*$	9_5

c

$7_3{}^*$	1_1	6_2
*	8_4	
$3_7{}^*$	4_6	9_5

a

$7_3{}^*$	1_1	6_2
	*	8_4
	9_5	

a

$7_3{}^*$	1_1	6_2
	9_5	
8_4	*	

b

$7_3{}^*$	1_1	6_2
3_5		
*	8_4	

a

$7_3{}^*$	1_1	6_2
	9_5	
	*	8_4

a

	1_1	2_4
	6_2	
3_5	$7_3{}^*$	*

a

	1_1	
	6_2	2_4
3_5	$7_3{}^*$	*

b

9_5	1_1	*
	6_2	
	$7_3{}^*$	2_4

b

	1_1	8_4
	6_2	
3_5	$7_3{}^*$	*

a

9_5	1_1	*
	6_2	8_4
	$7_3{}^*$	

a

9_5	1_1	*
	6_2	
	$7_3{}^*$	8_4

a

3_5	1_1	2_4
	$7_3{}^*$	
	6_2	*

a

3_5	1_1	
	$7_3{}^*$	2_4
	6_2	*

a

	1_1	
3_5	$7_3{}^*$	*
	6_2	2_4

a

5_5	1_1	4_4
	$7_3{}^*$	
	6_2	*

a

5_5	1_1	*
	$7_3{}^*$	4_4
	6_2	*

b

3_5	1_1	8_4
	$7_3{}^*$	
	6_2	*

a

9_5	1_1	*
	$7_3{}^*$	8_4
	6_2	

a

9_5	1_1	*
	$7_3{}^*$	
	6_2	8_4

Case II (Cont'd.)

a

3_3*	1_1	8_2
2_4	*	
	5_5	

a

3_3*	1_1	8_2
5_5		
*	2_4	

a

3_3*	1_1	8_2
4_4	*	
		7_5

a

3_3*	1_1	8_2
*	4_4	
7_5		

a

3_3*	1_1	8_2
*		4_4
7_5		

a

3_3*	1_1	8_2
	7_5	
4_4		*

a

3_3*	1_1	8_2
7_5		
*	4_4	

a

3_3*	1_1	8_2
7_5		
*		4_4

a

3_3*	1_1	8_2
6_4	*	
	9_5	

c

3_3*	1_1	8_2
	6_4	
2_6	7_7*	9_5

c

3_3*	1_1	8_2
*	6_4	
7_7*	2_6	9_5

c

3_3*	1_1	8_2
	6_4	
4_6	5_7*	9_5

c

3_3*	1_1	8_2
*	6_4	
5_7*	4_6	9_5

a

3_3*	1_1	8_2
	*	6_4
	9_5	

a

3_3*	1_1	8_2
	9_5	
6_4	*	

b

3_3*	1_1	8_2
7_5		
*	6_4	

a

3_3*	1_1	8_2
	9_5	
	*	6_4

a

	1_1	4_4
	8_2	
7_5	3_3*	*

a

	1_1	
	8_2	4_4
7_5	3_3*	*

b

9_5	1_1	*
	8_2	
	3_3*	4_4

b

	1_1	6_4
	8_2	
7_5	3_3*	*

a

9_5	1_1	*
	8_2	6_4
	3_3*	

a

9_5	1_1	*
	8_2	
	3_3*	6_4

a

5_5	1_1	2_4
	3_3*	
	8_2	*

a

5_5	1_1	*
	3_3*	2_4
	8_2	*

a

7_5	1_1	4_4
	3_3*	
	8_2	*

a

7_5	1_1	
	3_3*	4_4
	8_2	*

a

	1_1	
7_5	3_3*	*
	8_2	4_4

b

7_5	1_1	6_4
	3_3*	
	8_2	*

a

9_5	1_1	*
	3_3*	6_4
	8_2	

a

9_5	1_1	*
	3_3*	
	8_2	6_4

Consider the game
move three plays i
is shown in (*ii*). M
containing 2 by pl
block, ODD has th
seven (*iv*) or (*v*).

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where move five
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ODD's Winning
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where the digits 1-
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pp. 114-123 and 2
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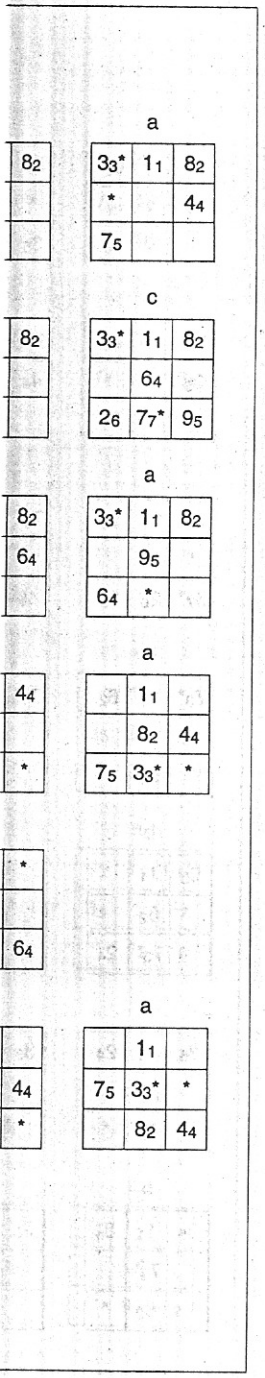

Consider the game shown in Example 4. The game starts out as in Example 1, but move three plays in C (*i*). Now, move four has several options and 8 played in D is shown in (*ii*). Move five forces EVEN to play a fatal number in the open line containing 2 by playing 5 in G (*iii*). Regardless of which number EVEN plays to block, ODD has the numbers available to complete the line for the win on move seven (*iv*) or (*v*).

Another example can be found by examining the games in the proof (Case II) where move five playing 5 has created a "double" force. Some of the games shown nearby could be won according to a similar "strategy." Several other examples illustrating other strategies could also be shown. However, of the strategies developed, the one presented here was the most compact and "neat" in terms of being able to describe.

Notice in Example 4 that ODD was required to play an "extra move" to win the game. By examining the proof, a pattern can be seen where when move two played a non-forcing number in three open lines, ODD could win on move five. If move two played a "forcing" number in three open lines (without forcing), ODD could win on move seven. Similar patterns are observed for other moves two and four. These observations tend to indicate that a conjecture is in order:

Conjecture

ODD's Winning Strategy (as presented in section 3) leads to the shortest possible win for ODD in each case determined by move two and/or move four.

The reader is challenged to prove or disprove this statement as well as to find yet a more compact strategy for ODD to win Graham's numerical tic-tac-toe.

Another question that might be investigated involves a variation of the game where the digits 1–9 are randomly divided between the two players. One player gets five numbers and plays first and the other player gets four numbers and plays second. The remaining rules of the game are the same. Is every game like this a first-player win?

Finally, this parting advice is offered to those who might engage a worthy adversary in a quick game or two. Don't play stupid, and if your opponent pulls out a magic square, don't even think of playing EVEN.

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