THE k-ZECKENDORF ARRAY

CURTIS COOPER

Abstract. Let $k \geq 2$ be an integer. We define the k-generalized Fibonacci sequence, the k-Zeckendorf representation of a positive integer, and the k-Zeckendorf array. When $k = 2$ these definitions are the Fibonacci sequence, the Zeckendorf representation of a positive integer, and the Zeckendorf array defined by Kimberling. The 3-Zeckendorf array is

1	2	4	7	13	24	44	81	149	274	504	⋯
3	6	11	20	37	68	125	230	423	778	1431	
5	9	17	31	57	105	193	355	653	1201	2209	
8	15	28	51	94	173	318	585	1076	1979	3640	
10	19	35	64	118	217	399	734	1350	2483	4567	
12	22	41	75	138	254	467	859	1580	2906	5345	
14	26	48	88	162	298	548	1008	1854	3410	6272	
16	30	55	101	186	342	629	1157	2128	3914	7199	
18	33	61	112	206	379	697	1282	2358	4337	7977	
⋮											

We prove that each of these k-Zeckendorf arrays is an interspersion.

1. Definitions

We begin with the definition of the k-generalized Fibonacci sequence $\{G_n\}$ for a fixed integer $k \geq 2$. This definition can be found in [1].

Definition 1.1. *Let $k \geq 2$ be an integer. The k-generalized Fibonacci sequence $\{G_n\}$ is defined as*

$$G_n = \begin{cases} 0, & \text{for } 0 \leq n < k-1 \\ 1, & \text{for } n = k-1 \\ \sum_{i=1}^{k} G_{n-i}, & \text{for } n \geq k. \end{cases}$$

Note that the Fibonacci sequence, $\{F_n\}$, is just the 2-generalized Fibonacci sequence. The first few terms of the k-generalized Fibonacci sequences for $2 \leq k \leq 8$ are given in the following table.

k-generalized Fibonacci Sequences

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136
4	0	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872
5	0	0	0	0	1	1	2	4	8	16	31	61	120	236	464	912	1793
6	0	0	0	0	0	1	1	2	4	8	16	32	63	125	248	492	976
7	0	0	0	0	0	0	1	1	2	4	8	16	32	64	127	253	504
8	0	0	0	0	0	0	0	1	1	2	4	8	16	32	64	128	255

The k-generalized Fibonacci sequences for $k = 2, 3, 4, 5, 6, 7, 8$ can be found in Sloane [8] as sequences A000045, A000073, A000078, A001591, A001592, A122189, and A079262, respectively.
We note that for each k-generalized Fibonacci sequence $\{G_n\}$,

$$G_n = 2^{n-k} \text{ for } k \le n \le 2k - 1.$$

We next define, for a fixed integer $k \ge 2$, the k-Zeckendorf representation of a positive integer. This definition is given in [7].

Definition 1.2. *Let $k \ge 2$ be an integer and let $\{G_n\}$ be the k-generalized Fibonacci sequence. The k-Zeckendorf representation of a positive integer n is defined as*

$$1d_{r-1} \cdots d_{k+1}d_k$$

where $n = \sum_{k \le i \le r} d_i G_i$, $d_i \in \{0, 1\}$ for all $k \le i \le r$, $d_r = 1$, and the product of any k consecutive d's is 0.

By use of the greedy algorithm, it can be seen that every integer has at least one representation of this form. However, by applying a theorem proved by Fraenkel that deals with unique representation in a numeration system defined by a sequence [2, Theorem 2], it follows that the k-Zeckendorf representation of a positive integer is unique. We list the 2-, 3-, 4-, and 5-Zeckendorf representations for $1 \le n \le 34$.

2-, 3-, 4-, and 5-Zeckendorf Representations

n	2-Zeckendorf	3-Zeckendorf	4-Zeckendorf	5-Zeckendorf
1	1	1	1	1
2	10	10	10	10
3	100	11	11	11
4	101	100	100	100
5	1000	101	101	101
6	1001	110	110	110
7	1010	1000	111	111
8	10000	1001	1000	1000
9	10001	1010	1001	1001
10	10010	1011	1010	1010
11	10100	1100	1011	1011
12	10101	1101	1100	1100
13	100000	10000	1101	1101
14	100001	10001	1110	1110
15	100010	10010	10000	1111
16	100100	10011	10001	10000
17	100101	10100	10010	10001
18	101000	10101	10011	10010
19	101001	10110	10100	10011
20	101010	11000	10101	10100
21	1000000	11001	10110	10101
22	1000001	11010	10111	10110
23	1000010	11011	11000	10111
24	1000100	100000	11001	11000
25	1000101	100001	11010	11001
26	1001000	100010	11011	11010
27	1001001	100011	11100	11011
28	1001010	100100	11101	11100
29	1010000	100101	100000	11101
30	1010001	100110	100001	11110
31	1010010	101000	100010	100000
32	1010100	101001	100011	100001
33	1010101	101010	100100	100010
34	10000000	101011	100101	100011

Now finally, we define the k-Zeckendorf array for a fixed integer $k \geq 2$.

Definition 1.3. *Let $k \geq 2$ be an integer. For all positive integers $j \geq 1$, the jth column of the k-Zeckendorf array is the increasing sequence of all positive integers whose least nonzero digit in its k-Zeckendorf representation is d_{j+k-1}.*

The 2-Zeckendorf array is just the Zeckendorf array in Kimberling [5]. This array is also the Wythoff array. We list it below.

Zeckendorf Array

1	2	3	5	8	13	21	34	55	89	144	⋯
4	7	11	18	29	47	76	123	199	322	521	
6	10	16	26	42	68	110	178	288	466	754	
9	15	24	39	63	102	165	267	432	699	1131	
12	20	32	52	84	136	220	356	576	932	1508	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
19	31	50	81	131	212	343	555	898	1453	2351	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

However for $k > 2$, the k-Zeckendorf arrays are different from Kimberling's k-order Zeckendorf arrays [5]. To construct the 3-Zeckendorf array, its first column is the increasing sequence of positive integers whose 3-Zeckendorf representations end with a 1, i.e., $1, 11, 101, 1001, 1011, 1101, 10001, 10011, 10101, \ldots$. Therefore, the first column is $1, 3, 5, 8, 10, 12, 14, 16, 18, \ldots$. The second column is the increasing sequence of positive integers whose 3-Zeckendorf representations end with 10, i.e. $10, 110, 1010, 10010, 10110, 11010, 100010, 100110, 101010, \ldots$. Therefore, the second column is $2, 6, 9, 15, 19, 22, 26, 30, 33, \ldots$. Continuing in this manner, the 3-Zeckendorf array is given below.

3-Zeckendorf Array

1	2	4	7	13	24	44	81	149	274	504	⋯
3	6	11	20	37	68	125	230	423	778	1431	
5	9	17	31	57	105	193	355	653	1201	2209	
8	15	28	51	94	173	318	585	1076	1979	3640	
10	19	35	64	118	217	399	734	1350	2483	4567	
12	22	41	75	138	254	467	859	1580	2906	5345	
14	26	48	88	162	298	548	1008	1854	3410	6272	
16	30	55	101	186	342	629	1157	2128	3914	7199	
18	33	61	112	206	379	697	1282	2358	4337	7977	
⋮											

The 3-Zeckendorf array can be found in Sloane [8] as sequence A136175. The 4-Zeckendorf array is listed below.

4-Zeckendorf Array

1	2	4	8	15	29	56	108	208	401	773	1490	⋯
3	6	12	23	44	85	164	316	609	1174	2263	4362	
5	10	19	37	71	137	264	509	981	1891	3645	7026	
7	14	27	52	100	193	372	717	1382	2664	5135	9898	
9	17	33	64	123	237	457	881	1698	3273	6309	12161	
11	21	41	79	152	293	565	1089	2099	4046	7799	15033	
13	25	48	93	179	345	665	1282	2471	4763	9181	17697	
16	31	60	116	223	430	829	1598	3080	5937	11444	22059	
18	35	68	131	252	486	937	1806	3481	6710	12934	24931	
⋮												

Finally, the 5-Zeckendorf array is listed below.

5-Zeckendorf Array

1	2	4	8	16	31	61	120	236	464	912	1793	⋯
3	6	12	24	47	92	181	356	700	1376	2705	5318	
5	10	20	39	77	151	297	584	1148	2257	4437	8723	
7	14	28	55	108	212	417	820	1612	3169	6230	12248	
9	18	35	69	136	267	525	1032	2029	3989	7842	15417	
11	22	43	85	167	328	645	1268	2493	4901	9635	18942	
13	26	51	100	197	387	761	1496	2941	6230	11367	22347	
15	30	59	116	228	448	881	1732	3405	6694	13160	25872	
17	33	65	128	252	495	973	1913	3761	7394	14536	28577	
⋮												

A few observations are obvious from the definition of the k-Zeckendorf array for every positive integer $k \geq 2$. First, every positive integer appears exactly once in the k-Zeckendorf array. Second, the first row of the k-Zeckendorf array is the k-generalized Fibonacci sequence starting at the kth term. In addition, upon examining the k-Zeckendorf arrays as k increases, the k-Zeckendorf array appears to approach a limiting array - perhaps A054582 in [8].

A054582

1	2	4	8	16	32	64	128	256	512	1024	2048	$\cdots$
3	6	12	24	48	96	192	384	768	1536	3072	6144	
5	10	20	40	80	160	320	640	1280	2560	5120	10240	
7	14	28	56	112	224	448	896	1792	3584	7168	14336	
9	18	36	72	144	288	576	1152	2304	4608	9216	18432	
11	22	44	88	176	352	704	1408	2816	5632	11264	22528	
13	26	52	104	208	416	832	1664	3328	6656	13312	26624	
15	30	60	120	240	480	960	1920	3840	7680	15360	30720	
17	34	68	136	272	544	1088	2176	4352	8704	17408	34816	
$\vdots$												

2. Shift Function

Let $\mathbb{N}$ denote the natural numbers. A helpful function is the shift function $f\colon \mathbb{N} \to \mathbb{N}$ for $k \geq 2$ an integer.

Definition 2.1. *Let $k \geq 2$ be an integer and $\{G_n\}$ be the k-generalized Fibonacci sequence. Define $f\colon \mathbb{N} \to \mathbb{N}$ as follows. Let n be a positive integer with k-Zeckendorf representation*

$$1d_{r-1} \cdots d_{k+1}d_k.$$

Then $f(n)$ is the positive integer whose k-Zeckendorf representation is

$$1d_{r-1} \cdots d_{k+1}d_k0.$$

Lemma 2.2. *The shift function f is a strictly increasing function.*

Proof. To prove the shift function is strictly increasing, we will show that if $m, n \in \mathbb{N}$ and $m < n$, then $f(m) < f(n)$.

The proof is by induction on $n \geq 2$.

The base step, $n = 2$, is true since for $1 < 2$, $f(1) = 2$ and $f(2) = G_{k+2}$ and $2 < G_{k+2}$.

For the induction step, assume $n \geq 2$ and for every $m < n$, $f(m) < f(n)$. Let m and $n+1$ have k-Zeckendorf representations

$$1d_{s-1} \cdots d_{k+1}d_k \quad \text{and} \quad 1e_{t-1} \cdots e_{k+1}e_k,$$

respectively.

Case 1. $s < t$. The largest number having an $(s-k+1)$-digit k-Zeckendorf representation can be found by writing

$$s - k + 1 = \left\lfloor \frac{s-k+1}{k} \right\rfloor k + r = q \cdot k + r,$$

where $0 \le r < k$. Let x be the number with k-Zeckendorf representation

$$\underbrace{\underbrace{11\cdots1}_{k-1}0\underbrace{11\cdots1}_{k-1}0\cdots\underbrace{11\cdots1}_{k-1}0}_{q}\underbrace{11\cdots1}_{r}.$$

Since $x+1 = G_{s+1}$, we have $m \le x < n+1$ and by the induction hypothesis, $f(m) \le f(x)$. But the k-Zeckendorf representation of $f(x)$ is

$$\underbrace{\underbrace{11\cdots1}_{k-1}0\underbrace{11\cdots1}_{k-1}0\cdots\underbrace{11\cdots1}_{k-1}0}_{q}\underbrace{11\cdots1}_{r}0,$$

which means that $f(x) < G_{s+2} \le G_{t+1} \le f(n+1)$.

Case 2. $s = t$. Here, $m - G_t < n+1-G_t$. By the induction hypothesis,

$$f(m-G_t) < f(n+1-G_t).$$

Then

$$f(m) = f(m-G_t) + G_{t+1} < f(n+1-G_t) + G_{t+1} = f(n+1).$$

In both cases, $f(m) < f(n+1)$ for all $m < n+1$. Thus, the induction step is true.

Therefore, by mathematical induction, the lemma is true.

□

Theorem 2.3. *Let $k \ge 2$ be an integer. The first column of the k-Zeckendorf array $Z = z(i,j)$ determines all of the array by the recurrences*

$$z(i, j+1) = f(z(i,j)) \tag{1}$$

for all $i \ge 1$ and $j \ge 1$.

Proof. The proof will be by induction on $i \ge 1$.

For the base step, $i = 1$, we have $z(1,j) = G_{j+k-1}$ for all $j \ge 1$, so that row 1 of Z is determined by $z(1,1) = 1$ and f.

For the induction step, assume $i \ge 1$ and that (1) holds for all $j \ge 1$ and for all $n \le i$. Write the k-Zeckendorf representation of $z(i+1,1)$ as

$$1d_{r-1}\cdots d_{k+1}1.$$

Let $m = f(z(i+1,1))$. Then the k-Zeckendorf representation of m has $r-k+2$ digits and they are

$$1d_{r-1}\cdots d_{k+1}10.$$

Also, m is in column 2 of Z, since its last two digits are 10. By the induction hypothesis, $z(j,2) = f(z(j,1))$ for $j = 1, 2, \ldots, i$, and since column 2 is an increasing sequence, m must lie in a row numbered $\ge i+1$ by Lemma 2.2. We shall show that this row number cannot be $> i+1$.

Suppose $m > z(i+1,2)$ and let the k-Zeckendorf representation of $z(i+1,2)$ be

$$e_s e_{s-1} \cdots e_{k+1} e_k = 1 e_{s-1} \cdots 10.$$

Then the number q having the k-Zeckendorf representation

$$e_s e_{s-1} \cdots e_{k+1} = 1 e_{s-1} \cdots 1$$

lies in column 1 of Z. It is not one of the first i terms, and it is not $z(i+1,1)$ since its sequel in row $i+1$ is not m. Therefore, $q = z(t,1)$ for some $t \geq i+2$. We now have $z(i+1,1) < q$ and $f(q) < f(z(i+1,1))$, a contradiction to Lemma 2.2. Therefore, $z(i+1,2) = f(z(i+1,1))$.

Let $j \geq 2$ and suppose that $z(i+1,j) = f(z(i+1,j-1))$. The argument just used for $j=2$ applies here in the same way, giving $z(i+1,j+1) = f(z(i+1,j))$. The induction on j is finished, so that (1) holds for all $j \geq 1$ and $i+1$. Consequently, the induction on i is finished, so that (1) holds throughout Z.

□

3. Interspersion and Other Properties

Some of the material in this section overlaps with Kimberling's [4], in particular Kimberling's proof of Lemma 3.1 and Theorem 3.

The definition of an interspersion was introduced by Kimberling [3].

Definition 3.1. *An array $A = a(i,j)$ of positive integers is called an interspersion if*

1. *The rows of A comprise a partition of the positive integers,*
2. *Every row of A is an increasing sequence,*
3. *Every column of A is a (possibly finite) increasing sequence,*
4. *If $\{u_j\}$ and $\{v_j\}$ are distinct rows of A and if p and q are any indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$.*

Theorem 3.2. *For every $k \geq 2$, the k-Zeckendorf array is an interspersion.*

Proof. Let Z be the k-Zeckendorf array for a positive integer $k \geq 2$. We have shown that every positive integer occurs exactly once in Z and that every column of Z is increasing. Because of Theorem 2.3, every row of Z is increasing. To prove the fourth property, suppose i, j, i', j' are indices for which

$$z(i,j) < z(i',j') < z(i,j+1).$$

Then, by Lemma 2.2,

$$f(z(i,j)) < f(z(i',j')) < f(z(i,j+1)),$$

so that, by Theorem 2.3,

$$z(i,j+1) < z(i',j'+1) < z(i,j+2).$$

Therefore, the fourth property of an interspersion is satisfied. Therefore, Z is an interspersion. □

Here are some other properties of the k-Zeckendorf array.

Lemma 3.3. *Let $k \geq 2$ be a positive integer. Then the leading term in each row of the k-Zeckendorf array is the smallest number not found in any earlier row.*

Proof. Let $k \geq 2$ be a positive integer and let Z be the k-Zeckendorf array. Let y be a positive integer in column 1 and row greater than 1 of Z. By the definition of Z, the k-Zeckendorf representation of y is

$$1d_{r-1}\cdots d_{k+1}1,$$

where $r > 2$. Let x be a positive integer and $x < y$. We will show that x is in an earlier row than y.

Case 1. The k-Zeckendorf representation of x ends in a 1. Then by the definition of Z, x is in column 1 of Z and since $x < y$, x is in an earlier row of Z than y.

Case 2. The k-Zeckendorf representation of x ends in a 0. Let x' be the positive integer whose k-Zeckendorf representation is the one obtained by deleting all the 0's at the end of the k-Zeckendorf representation of x. Then x' ends in a 1 and $x' < y$ so by the definition of Z, x' is in an earlier row of Z than y. But x is in the same row as x' so x is in an earlier row of Z than y.

This completes the proof. □

Lemma 3.4. *Let $k \geq 2$ be a positive integer. Every row of the k-Zeckendorf array satisfies the k-generalized Fibonacci recurrence.*

Proof. Let $k \geq 2$ be a positive integer. Let Z be the k-Zeckendorf array. Consider any k consecutive elements in a row of the k-Zeckendorf array, say

$$z(i,j),\ z(i,j+1),\ \ldots, z(i,j+k-1),$$

where i and j are positive integers. In addition, let the k-Zeckendorf representation of $z(i,j)$ be

$$1d_{r-1}\cdots d_{k+1}d_k.$$

Then

$$z(i,j) = \sum_{k\leq n\leq r} d_n G_n.$$

By Theorem 2.3, for $1 \leq m \leq k-1$,

$$z(i,j+m) = f(z(i,j+m-1)),$$

where f is the shift function. Hence,

$$z(i,j+m) = \sum_{k\leq n\leq r} d_n G_{n+m}.$$

Therefore,

$$\begin{aligned}\sum_{0\leq m\leq k-1} z(i,j+m) &= \sum_{0\leq m\leq k-1}\sum_{k\leq n\leq r} d_n G_{n+m}\\ &= \sum_{k\leq n\leq r} d_n \sum_{0\leq m\leq k-1} G_{n+m}\\ &= \sum_{k\leq n\leq r} d_n G_{n+k}\\ &= z(i,j+k).\end{aligned}$$

Thus, the sum of any k consecutive elements in a row of the k-Zeckendorf array is the next element in the row. This completes the proof. □

4. Open Question

We have the following question we were not able to prove or disprove.

Question 4.1. *Does every positive k-generalized Fibonacci-type sequence (satisfying $a(n) = a(n-1) + \cdots + a(n-k)$ and eventually positive) appear as some row of the k-Zeckendorf array?*

In [6], Morrison proved the affirmative answer to Question 4.1 for the case $k = 2$. The question remains open for $k \geq 3$.

Acknowledgment

The author would like to thank an anonymous referee for some valuable suggestions that helped improve the content of the paper.

References

[1] C. Cooper, *An Identity for Generalized Fibonacci Numbers*, Journal of Institute of Mathematics & Computer Sciences, **20** (2007), 1–8.
[2] A. S. Fraenkel, *Systems of Numeration*, American Mathematical Monthly, **92** (1985), 105–114.
[3] C. Kimberling, *Interspersions and Dispersions*, Proc. Amer. Math. Soc., **117** (1993), 313–321.
[4] C. Kimberling, *Numeration Systems and Fractal Sequences*, Acta Arth., **73.2** (1995), 103–117.
[5] C. Kimberling, *The Zeckendorf Array Equals the Wythoff Array*, The Fibonacci Quarterly, **33.1** (1995), 3–8.
[6] D. R. Morrison, *A Stolarsky Array of Wythoff Pairs*, in A Collection of Manuscripts Related to the Fibonacci Sequence, pp. 134–136, The Fibonacci Association, Santa Clara, CA, 1980.
[7] A. Ray and C. Cooper, *On the Natural Density of the k-Zeckendorf Niven Numbers*, Journal of Institute of Mathematics & Computer Sciences, **19** (2006), 83–98.
[8] N. J. A. Sloane (Ed.), *The On-Line Encyclopedia of Integer Sequences (2008)*, published electronically at http://www.research.att.com/∼njas/sequences/.

AMS Classification Numbers: 11B39

Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, MO 64093, U.S.A.
E-mail address: cooper@ucmo.edu