# 1996 Missouri MAA Collegiate Mathematics Competition

## Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at $P$ will intersect the parabola at another point, say $Q$. Find the coordinates of $P$ so that the area bounded by the normal line and the parabola is a minimum.

2. If

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots ,$$

$$v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots ,$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots ,$$

prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

3. Each of the numbers $x_1, x_2, \ldots , x_n$ can be 1, 0, or -1. What is the minimum possible value of the sum of all products of pairs of these numbers?

4. A $6 \times 6$ board is tiled with $2 \times 1$ dominos. Prove that the board can be cut into two parts by a straight line that does not cut dominos.

5. Let $a, b, c, d, e$ be integers such that $1 \leq a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16},$$

where $[m,n]$ denotes the least common multiple of $m$ and $n$ (e.g. $[4,6] = 12$).

## Session II

1. Evaluate the definite integrals

(a)

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}},$$

(b)

$$\int_1^\infty \frac{dx}{e^{x+1}+e^{3-x}}.$$

2. Let $x, y, z$ be three different integers. Prove that

$$(x-y)^5 + (y-z)^5 + (z-x)^5$$

is divisible by $5(x-y)(y-z)(z-x)$.

3. What is the probability of an odd number of sixes turning up in a random toss of $n$ fair dice?

4. A swimmer stands at one corner of a square swimming pool and wishes to reach the diagonally opposite corner. If $w$ is the swimmer's walking speed and $s$ is the swimmer's swimming speed ($s < w$), find the swimmer's path for shortest time. [Consider two cases: (i) $w/s < \sqrt{2}$, and (ii) $w/s > \sqrt{2}$.]

5. In a finite sequence of real numbers, the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.