1996 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the area bounded by the normal line and the parabola is a minimum.

Solution.

Let P have the coordinates (p, p^2) and Q have the coordinates (q, q^2). Without loss of generality, let $p > 0$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = \frac{-1}{2p}(x - p).$$

Therefore, the point Q, the intersection of the normal line and the parabola has the x coordinate

$$q = -p - \frac{1}{2p}.$$

Hence, the area bounded by the normal line and the parabola is

$$A = \int_q^p \left(p^2 - \frac{1}{2p}(x-p) - x^2\right) dx = \int_q^p (p - x)\left(p + x + \frac{1}{2p}\right) dx.$$

Using the generalized Leibniz rule to differentiate the integral with respect to p yields

$$\frac{dA}{dp} = 0 - 0 + \int_q^p \left(2p + \frac{x}{2p^2}\right) dx = 2p(p-q) + \frac{p^2 - q^2}{4p^2}.$$

Setting this to 0, factoring out $p - q$ and substituting $-1/2p = p + q$ gives $p = 1/2$ as the minimum. Thus,

$$P = \left(\frac{1}{2}, \frac{1}{4}\right).$$

From the Putnam Exam, 1939, 14(i)

2. If

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots ,$$

$$v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots ,$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots ,$$

prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

Solution.

The power series for u, v, and w converge for all x, and

$$\frac{du}{dx} = w, \quad \frac{dv}{dx} = u, \quad \frac{dw}{dx} = v,$$

as we see by differentiating them. Letting

$$f = u^3 + v^3 + w^3 - 3uvw,$$

we have

$$f' = 3u^2u' + 3v^2v' + 3w^2w' - 3uvw' - 3uv'w - 3u'vw$$

$$= 3u^2w + 3v^2u + 3w^2v - 3uv^2 - 3u^2w - 3vw^2 = 0.$$

Thus $f =$ constant. But $f(0) = (u(0))^3 = 1$, so $f(x) = 1$ for all x.

From the All-Soviet Union Mathematics Competition, Moscow, 1965, 056

3. Each of the numbers $x_1, x_2, \ldots, x_n$ can be 1, 0, or -1. What is the minimum possible value of the sum of all products of pairs of these numbers?

Solution.

Let

$$S_n = \sum_{1 \le i < j \le n} x_i x_j = \frac{\left(\sum_{1 \le i \le n} x_i\right)^2 - \sum_{1 \le i \le n} x_i^2}{2}.$$

To minimize the last expression, we will consider two cases.

Case 1. n is even. S_n can be minimized by choosing half $(n/2)$ of the x_i's to be 1 and the other half $(n/2)$ of the x_i's to be -1. The minimum S_n is $-n/2$

Case 2. n is odd. S_n can be minimized by choosing $(n-1)/2$ of the x_i's to be 1 and $(n-1)/2$ of the x_i's to be -1. The value of the one left over has no effect on S_n. The minimum S_n is $-(n-1)/2$.

From the All-Soviet Union Mathematics Competition, Moscow, 1963, 033

4. A 6×6 board is tiled with 2×1 dominos. Prove that the board can be cut into two parts by a straight line that does not cut dominos.

Solution.

Suppose the 6×6 tiled board cannot be cut by a straight line without cutting a domino. Thus for the 5 horizontal and 5 vertical lines, there are 10 dominos which are cut by the 10 lines, e.g. the domino cut by the second vertical line is denoted with a horizontal line segment and the region to the left of the vertical line is marked with an A and the region to the right of the vertical line is marked with a B.

But the A area has an odd number of squares and thus the tiling must have another domino which is cut by the second vertical line. Likewise, each of the 10 lines cuts 2 dominos, for a total of 20 dominos. But this is a contradiction, since the tiling consists of 18 dominos.

From the 11th Canadian Mathematics Olympiad, 1979, 3

5. Let a, b, c, d, e be integers such that $1 \le a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \le \frac{15}{16},$$

where $[m, n]$ denotes the least common multiple of m and n (e.g. $[4, 6] = 12$).

Solution.

Let S denote the sum on the left of the proposed inequality. Since $1 \le a < \cdots < e$, we have

$$[a,b] \ge 2a, \quad [b,c] \ge 2b, \quad [c,d] \ge 2c, \quad [d,e] \ge 2d,$$

and $b \ge 2$, $c \ge 3$. We consider two cases:

Case 1. $c = 3$. If $d = 4$, then $[a,b] = 2$, $[b,c] = 6$, $[c,d] = 12$, and $[d,e] \ge 8$; hence

$$S \le \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{8} = \frac{7}{8} < \frac{15}{16}.$$

If $d \ge 5$, then $[c,d] \ge 6$ and $[d,e] \ge 10$; hence

$$S \le \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{10} = \frac{14}{15} < \frac{15}{16}.$$

Case 2. $c \ge 4$. If $5 \le d \le 7$, then $[c,d] = 20, 28, 30, 35$, or 42, except when $c = 4$ and $d = 6$. In the exceptional case, we have

$$S \le \frac{1}{2} + \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{11}{12} < \frac{15}{16};$$

and otherwise

$$S \le \frac{1}{2} + \frac{1}{4} + \frac{1}{20} + \frac{1}{10} = \frac{9}{10} < \frac{15}{16}.$$

Finally, if $d \ge 8$, we have

$$S \le \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

Session II

From the Putnam Exam, 1939, 9 (i) and (ii)

1. Evaluate the definite integrals

(a)

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}},$$

(b)

$$\int_1^\infty \frac{dx}{e^{x+1}+e^{3-x}}.$$

Solution.

(a) Since the integrand is not defined at either bound of integration, one should write

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \lim_{\substack{\epsilon\to 0^+\\ \delta\to 0^+}} \int_{1+\epsilon}^{3-\delta} \frac{dx}{\sqrt{(x-1)(3-x)}}$$

$$= \lim_{\substack{\epsilon\to 0^+\\ \delta\to 0^+}} \int_{1+\epsilon}^{3-\delta} \frac{dx}{\sqrt{1-(x-2)^2}} = \lim_{\substack{\epsilon\to 0^+\\ \delta\to 0^+}} \arcsin(x-2)\Big|_{1+\epsilon}^{3-\delta}$$

$$= \lim_{\substack{\epsilon\to 0^+\\ \delta\to 0^+}} [\arcsin(1-\delta) - \arcsin(\epsilon-1)] = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

(b) The difficulty here is with the infinite interval of integration. Let $y = x-1$; then

$$\int \frac{dx}{e^{x+1}+e^{3-x}} = \frac{1}{e^2}\int \frac{dx}{e^{x-1}+e^{1-x}} = \frac{1}{e^2}\int \frac{dy}{e^y+e^{-y}}$$

$$= \frac{1}{e^2}\int \frac{e^y\,dy}{e^{2y}+1} = \frac{1}{e^2}\arctan e^y + c.$$

Hence

$$\begin{aligned}\int_1^{\infty} \frac{dx}{e^{x+1}+e^{3-x}} &= \lim_{N\to\infty} \int_1^{N} \frac{dx}{e^{x+1}+e^{3-x}} \\ &= \lim_{N\to\infty} \frac{1}{e^2}[\arctan e^{N-1} - \arctan e^0] \\ &= \frac{1}{e^2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4e^2}.\end{aligned}$$

From the All-Soviet Union Mathematics Competition, Moscow, 1962, 024

2. Let x, y, z be three different integers. Prove that

$$(x-y)^5 + (y-z)^5 + (z-x)^5$$

is divisible by $5(x-y)(y-z)(z-x)$.

Solution.

$$\begin{aligned}&(x-y)^5 + (y-z)^5 + (z-x)^5 \\ &= (x-z+z-y)^5 + (y-z)^5 + (z-x)^5 \\ &= (x-z)^5 + 5(x-z)^4(z-y) + 10(x-z)^3(z-y)^2 + 10(x-z)^2(z-y)^3 \\ &\quad + 5(x-z)(z-y)^4 + (z-y)^5 + (y-z)^5 + (z-x)^5 \\ &= 5(x-z)^4(z-y) + 10(x-z)^3(z-y)^2 + 10(x-z)^2(z-y)^3 + 5(x-z)(z-y)^4 \\ &= 5(x-z)(z-y)\left((x-z)^3 + 2(x-z)^2(z-y) + 2(x-z)(z-y)^2 + (z-y)^3\right).\end{aligned}$$

But,

$$(x-z)^3+2(x-z)^2(z-y)+2(x-z)(z-y)^2+(z-y)^3$$

$$=(x-y+y-z)^3+2(x-y+y-z)^2(z-y)+2(x-y+y-z)(z-y)^2+(z-y)^3$$

$$=(x-y)^3+3(x-y)^2(y-z)+3(x-y)(y-z)^2+(y-z)^3+2(x-y)^2(z-y)$$

$$+4(x-y)(y-z)(z-y)+2(y-z)^2(z-y)+2(x-y)(z-y)^2+2(y-z)(z-y)^2+(z-y)^3$$

$$=(x-y)^3+3(x-y)^2(y-z)+3(x-y)(y-z)^2+2(x-y)^2(z-y)$$

$$+4(x-y)(y-z)(z-y)+2(x-y)(z-y)^2$$

has a factor of $x-y$. Therefore, the result follows.

From Crux Mathematicorum, 1979, Practice Set 4-1

3. What is the probability of an odd number of sixes turning up in a random toss of n fair dice?

Solution.

For $0 \le k \le n$, the probability of k sixes turning up in a random toss of n fair dice is

$$\binom{n}{k}\left(\frac{5}{6}\right)^{n-k}\left(\frac{1}{6}\right)^{k};$$

hence, with $a=5/6$ and $b=1/6$, the required probability is

$$P=\binom{n}{1}a^{n-1}b+\binom{n}{3}a^{n-3}b^3+\binom{n}{5}a^{n-5}b^5+\cdots$$

$$=\text{ sum of the even-ranked terms in the expansion of } (a+b)^n$$

$$=\frac{1}{2}\{(a+b)^n-(a-b)^n\}$$

$$=\frac{1}{2}\left\{1-\left(\frac{2}{3}\right)^n\right\}.$$

From the Putnam Exam, 1938, 6

4. A swimmer stands at one corner of a square swimming pool and wishes to reach the diagonally opposite corner. If w is the swimmer's walking speed and s is the swimmer's swimming speed ($s < w$), find the swimmer's path for shortest time. [Consider two cases: (i) $w/s < \sqrt{2}$, and (ii) $w/s > \sqrt{2}$.]

Solution.

Let the square pool be denoted by $ABCD$, with the swimmer initially at A and desirous of reaching C. The path of least time can evidently be described as follows. The swimmer walks from A to E (a point on side AB), swims from E to F where F is on BC, and then walks from F to C. Note that a path like $AGHC$ is time equivalent to a path of the type described with $F = C$.

Let $\overline{AE} = x$, $\overline{EF} = y$, $\overline{FC} = z$. Then the time T is given by $T = (x+z)/w + (y/s)$. If the sum $x + z$ is fixed, then the sum $y \sin\alpha + y \cos\alpha$ is also fixed, and y is minimal when $(\sin\alpha + \cos\alpha)$ is maximal. This maximum is attained for $\alpha = 45°$.

Thus for a minimal time path, $x = z$ and $y = \sqrt{2}(l - x)$, where l is the length of a side of the pool. Accordingly, we have to minimize $T = (2x/w) + \sqrt{2}(l - x)/s$ for $0 \le x \le l$.

But T is a linear function of x, so its maximum occurs at an endpoint of the interval. If $x = 0$, $T = \sqrt{2}l/s$, and if $x = l$, $T = 2l/w$.

If $\sqrt{2}l/s < 2l/w$ then $w/s < \sqrt{2}$, and conversely. Hence, if $w/s < \sqrt{2}$ the minimal path is unique and the swimmer should swim diagonally across the pool from A to C. If $w/s > \sqrt{2}$, the swimmer should walk from A to B to C. Finally, if $w/s = \sqrt{2}$, T is independent of x and there are infinitely many minimizing paths, in fact any path $AEFC$ for which $\alpha = 45°$.

From the International Mathematical Olympiad, 1977, 2

5. In a finite sequence of real numbers, the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

Solution.

Denote by S_n a sequence of n terms having the desired property. We shall prove that there is no S_{17}. Then we shall construct an S_{16} which automatically furnishes also S_n for $n < 16$. This will demonstrate that 16 is the desired maximum number of terms.

We denote the terms of S_{17} by a_1, a_2, $\ldots$, a_{17} and write all sets of seven consecutive terms in rows of the table below:

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \end{array}.$$

We note that the columns in this table contain all sets of eleven consecutive terms of S_{17}. Now we add all entries of the table first by adding all entries in each row, then by adding the row sums. By hypothesis, each row sum is negative, hence the total is also negative. Then we add all entries of the table by adding the entries in each column and then adding the column sums. By hypothesis each column sum is positive, hence the total is positive, which contradicts what we just found by adding the rows. We conclude that there is no S_{17}.

An example of S_{16} is the sequence

$$5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5,$$

and any n consecutive terms of S_{16} lead to a sequence S_n for $11 \leq n \leq 16$. Thus to obtain an S_{14}, just pick any 14 consecutive terms of S_{16}.

We shall now show how the above S_{16} can be found by a method other than guessing.

Assume we can find a sequence S_{16} that reads the same from left to right as from right to left, and that the sum of any seven consecutive terms is -1 and the sum of any eleven consecutive terms is 1. Then

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 &= -1 \\ a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 &= -1 \\ a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_8 &= -1 \\ a_4 + a_5 + a_6 + a_7 + a_8 + a_8 + a_7 &= -1 \\ a_5 + a_6 + a_7 + a_8 + a_8 + a_7 + a_6 &= -1. \end{aligned}$$

Subtracting the second equation from the first, then the third from the second, etc., we obtain

$$a_1 = a_8, \quad a_2 = a_8, \quad a_3 = a_7, \quad a_4 = a_6.$$

Now writing the sum of eleven consecutive terms

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_8 + a_7 + a_6 &= 1 \\ a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_8 + a_7 + a_6 + a_5 &= 1 \\ a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_8 + a_7 + a_6 + a_5 + a_4 &= 1 \end{aligned}$$

and subtracting the second equation from the first and the third from the second, we obtain

$$a_1 = a_5 \quad \text{and} \quad a_2 = a_4.$$

It follows that S_{16} has the form

$$a_1, a_1, a_3, a_1, a_1, a_1, a_3, a_1, a_1, a_3, a_1, a_1, a_1, a_3, a_1, a_1,$$

where the sum of any seven consecutive terms is $5a_1 + 2a_3 = -1$ and the sum of any eleven consecutive terms is $8a_1 + 3a_3 = 1$. The solution of this pair of equations is $a_1 = 5$, $a_3 = -13$ and leads to the S_{16} above.