1998 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the length of segment PQ is a minimum.

Solution.

Let $P = (p, p^2)$ and $Q = (q, q^2)$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Therefore, the other intersection point of the normal line and the parabola, Q has the x-coordinate

$$q = -p - \frac{1}{2p}.$$

Let

$$d = \left(2p + \frac{1}{2p}\right)^2 + \left(p^2 - \left(p^2 + 1 + \frac{1}{4p^2}\right)\right)^2$$

represent the square of the distance between P and Q. Differentiating d with respect to p and setting this expression to zero results in the equation

$$8p - \frac{3}{2}p^{-3} - \frac{1}{4}p^{-5} = 0.$$

Simplifying this equation, we obtain

$$\frac{1}{4}p^{-5}(32p^6 - 6p^2 - 1) = 0.$$

But,

$$32p^6 - 6p^2 - 1 = (2p^2 - 1)(4p^2 + 1)^2$$

so the real solutions of the equation are $p = \pm\sqrt{1/2}$. Hence, the coordinates of P so that the length of segment PQ is a minimum is

$$P = \left(\pm\sqrt{\frac{1}{2}}, \frac{1}{2}\right).$$

2. Let $0 < a_1 < a_2 < \cdots < a_n$ and let $e_i = \pm 1$. Prove that $\sum_{i=1}^{n} e_i a_i$ assumes at least $\binom{n+1}{2}$ distinct values as the e_i's range over the 2^n possible combinations of signs.

Solution.

The result is clear if $n = 1$. Assume the result is correct for some n and suppose $0 < a_1 < a_2 < \cdots < a_n < a_{n+1}$. By the induction assumption,

$$\sum_{i=1}^{n} e_i a_i + a_{n+1}$$

assumes at least $\binom{n+1}{2}$ distinct values, the smallest of which is

$$-\sum_{i=1}^{n} a_i + a_{n+1}.$$

Let

$$s_p = -\sum_{i=1}^{p-1} a_i + a_p - \sum_{i=p+1}^{n+1} a_i$$

for $p = 0, 1, 2, \ldots, n, n+1$. Note that

$$s_{n+1} = -\sum_{i=1}^{n} a_i + a_{n+1}.$$

Also, note that s_p is strictly increasing. Therefore, the $n+1$ additional values s_p, $p = 0, 1, 2, \ldots, n$ are all smaller than s_{n+1}. It follows that

$$\sum_{i=1}^{n+1} e_i a_i$$

assumes at least $\binom{n+1}{2} + (n+1) = \binom{n+2}{2}$ distinct values.

3. If m and n are positive integers and $a < b$, find a formula for

$$\int_a^b \frac{(b-x)^m}{m!} \frac{(x-a)^n}{n!} dx$$

and use your formula to evaluate

$$\int_0^1 (1-x^2)^n dx.$$

Solution.
Integration by parts once yields

$$\int_a^b \frac{(b-x)^m}{m!} \frac{(x-a)^n}{n!} dx = \int_a^b \frac{(b-x)^{m-1}}{(m-1)!} \frac{(x-a)^{n+1}}{(n+1)!} dx$$

so continuing, we get

$$\int_a^b \frac{(b-x)^m}{m!} \frac{(x-a)^n}{n!} dx = \int_a^b \frac{(x-a)^{n+m}}{(n+m)!} dx = \frac{(b-a)^{n+m+1}}{(n+m+1)!}.$$

Now,

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2} \int_{-1}^1 (1-x)^n (x-(-1))^n dx = \frac{1}{2} (n!)^2 \frac{2^{2n+1}}{(2n+1)!} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}.$$

4. Describe how to fold a rectangular sheet of paper so that the lower right corner touches the left edge and the length of the crease is a minimum. Discuss how the dimensions of the rectangle affect the result.

Solution.

Let the width and the height of the page be given by w and h and the length of the crease by c. Assume $h > w$, because otherwise the minimum is clearly h when the paper is folded vertically in half. There are three remaining cases: (i) when the crease extends from the bottom edge to the top edge, (ii) when the crease extends from the bottom edge to the right edge, and (iii) when the crease extends from the left edge to the right edge. In case (i) the minimum is clearly when the crease is vertical with $c = h$. Also clear is case (iii), in which the minimum occurs when the lower right corner is placed on the upper left corner. Here

$$c = \frac{w}{h}\sqrt{w^2 + h^2}.$$

In case (ii), let a and b be the short leg and the long leg of the right triangle whose hypotenuse is c and whose right angle is the corner which was folded over to the left edge. Let x be the distance from this corner (after the fold) to the lower left corner of the page. From the Pythagorean Theorem,

$$x^2 + (w - a)^2 = a^2 \text{ and } w^2 + (b - x)^2 = b^2.$$

Solving these equations for a and b yields

$$a = \frac{x^2 + w^2}{2w} \text{ and } b = \frac{x^2 + w^2}{2x}.$$

Now, since $c^2 = a^2 + b^2$, we have c^2 in terms of x. Differentiation yields a relative minimum at $x = w/\sqrt{2}$, where $c = \frac{3\sqrt{3}}{4}w$. The minimum is therefore

$$\min\left\{h, \frac{3\sqrt{3}}{4}w, \frac{w}{h}\sqrt{w^2 + h^2}\right\},$$

but we can eliminate the case (ii) value because that leads to the inequality

$$\frac{3\sqrt{3}}{4} < \frac{h}{w} < \frac{4\sqrt{11}}{11}$$

which is impossible because $\frac{3\sqrt{3}}{4} > \frac{4\sqrt{11}}{11}$. The minimum is h (case (i)) whenever

$$0 < \frac{w}{h} < \sqrt{\frac{\sqrt{5}-1}{2}}$$

and

$$\frac{w}{h}\sqrt{w^2 + h^2}$$

(case (iii)) otherwise.

Modified from the 1967 All-Soviet Mathematics Competition - Problem 88

5. Let n be a positive integer. Prove that there exists a number divisible by 5^n that does not contain a single zero in its decimal notation.

Solution.

Perform the following algorithm.

Algorithm.

$m := 5^n$;
for $i := 0$ **to** n
  if the digit in the 10^ith place of m is 0
    then $m := m + 10^i \cdot 5^n$;

The resulting m has no zero from its 10^0th to 10^nth places and is a multiple of 5^n.

Next, change each zero in m to any nonzero digit and call this number p. Note that any zero in m must be in its 10^{n+1}th place or higher.

The number p is a multiple of 5^n and does not contain a single zero in its decimal notation.

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Session II

1. Let I be the $n \times n$ identity matrix. Prove that $AB - BA \neq I$ for any $n \times n$ matrices A and B.

Solution.

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. Then the sum of the diagonal (i.e., the trace) of $AB - BA$ is

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{ji} - \sum_{j=1}^{n} b_{ij}a_{ji}\right) = 0.$$

Therefore, $AB - BA \neq I$.

2. Let $a_1, \ldots, a_n$ be positive real numbers and let s denote their sum. Show that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \leq 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!}.$$

Solution.

By the Arithmetic Mean-Geometric Mean inequality,

$$\left(\prod_{j=1}^{n}(1+a_j)\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{j=1}^{n}(1+a_j) = \frac{1}{n}(n+s) = 1 + \frac{s}{n},$$

so

$$\prod_{j=1}^{n}(1+a_j) \leq \left(1+\frac{s}{n}\right)^n = \sum_{j=0}^{n}\binom{n}{j}\frac{s^j}{n^j} = \sum_{j=0}^{n}\frac{n!}{(n-j)!\cdot n^j}\cdot\frac{s^j}{j!} \leq \sum_{j=0}^{n}\frac{s^j}{j!}.$$

From *Crux Mathematicorum*, vol. 8 (1982), p. 99 and pp. 271–272.

Proposed by Jack Brennen, student, Poolesville, Maryland.

3. Sum the series

$$\sum_{i=1}^{\infty} \frac{36i^2+1}{(36i^2-1)^2}.$$

Solution.

If S is the required sum, then we have

$$2S = \sum_{i=1}^{\infty} \left(\frac{1}{(6i-1)^2} + \frac{1}{(6i+1)^2} \right) = \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots .$$

Now, using the result

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

we obtain successively

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24},$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8},$$

$$\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \cdots = \frac{\pi^2}{72},$$

and

$$2S + 1 = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots = \frac{\pi^2}{8} - \frac{\pi^2}{72} = \frac{\pi^2}{9},$$

from which

$$S = \frac{\pi^2 - 9}{18}.$$