2001 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the area bounded by the normal line and the parabola is a minimum.

2. Let $\{x_i\}$ denote any finite sequence with the following properties:

(a) $x_i \in \{-2, 1, 2\}$ for each x_i,
(b) $\sum_i x_i = 29$,
(c) $\sum_i x_i^2 = 59$.

In considering the family of all such sequences, let $M = \max\{\sum_i x_i^3\}$ and $m = \min\{\sum_i x_i^3\}$. Determine M/m.

3. Let a, b, and c be the sides of a triangle with perimeter 2. Prove that

$$3/2 < a^2 + b^2 + c^2 + 2abc < 2.$$

4. Find the sum

$$S = \sum_{k=1}^{\infty} \frac{k^2}{3^k}.$$

5. A set of five cubical dice has the following properties:

(a) On each face of each die is a 3-digit integer. No two integers on a given face are the same.
(b) Every integer has a nonzero hundred's digit.
(c) The sum, when the dice are rolled, of the five integers is a 4-digit integer.
(d) Whenever the dice are rolled, their sum S can be found quickly as follows: the sum of the unit's digits of the five dice is the last two digits of S, and the first two digits of S are 50 minus the sum of the unit's digits.

For example, if the dice come up 189, 256, 275, 845, and 168, the sum of the unit's digits is $9 + 6 + 5 + 5 + 8 = 33$, so the value of S is 1733, since $50 - 33 = 17$.

Explain, justifying your statements, how such a set of dice can be constructed.

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Session II

1. Circle B lies wholly in the interior of circle A. Find the loci of points equidistant from the two circles?

2. Show that if x, y, and z are positive reals such that $x+y+z=1$, then

$$\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \geq 8.$$

3. A convex decagon and all of its diagonals are drawn. How many *interior* points of intersection of the diagonals are there, if it is assumed that no 3 diagonals share a common *interior* point?

4. No matter what n real numbers x_1, x_2, $\ldots$, x_n may be selected in the closed unit interval $[0,1]$, prove that there always exists a real number x in this interval such that the average unsigned distance from x to the x_i's is exactly $1/2$.

5. Consider the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1,$$

where $a_i \geq 0$. If the equation $f(x) = 0$ happens to have n real roots, is it not remarkable that the value of $f(2)$ must then be at least 3^n? Prove this unlikely consequence: $f(2) \geq 3^n$.