2003 Missouri Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the sum of the y-coordinates (or the sum of the ordinates) of P and Q is a minimum.

2. There is an $m \times n$ rectangular array of office mailboxes in which mail for $p \leq mn$ people is distributed. Initially, the mailboxes are assigned alphabetically beginning at the upper left and proceeding down each column (the "next" mailbox to one at the bottom of a column is the one at the top of the next column to the right). A new secretary is hired, and decides that the mailboxes will now be assigned alphabetically beginning at the upper left and proceeding to the right across each row. Discuss, as completely as possible, whose mailboxes will be unchanged.

3. For sufficiently small but positive θ, the relation $\tan\theta > \theta$ holds. Prove, in the other direction, that for $0 < \theta < \pi/4$ one has

$$\tan\theta < \frac{4\theta}{\pi}.$$

4. Let $ABCD$ be a quadrilateral, with sides $AB = a$, $BC = b$, $CD = c$, where a, b, and c are fixed positive quantities. Prove that when the quadrilateral $ABCD$ has a maximum area, then $ABCD$ can be inscribed in a semicircle.

5. Define a sequence $\{x_n\}_{n=2}^{\infty}$ by

$$(n + x_n)[\sqrt[n]{2} - 1] = \ln 2.$$

Find $\lim_{n\to\infty} x_n$.

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Session II

1. Suppose that a, b and c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}.$$

2. Let $x_1 > 1$ be odd and define the sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_n = x_{n-1}^2 - 2$, $n \geq 2$. Prove that for any pair of integers j, k satisfying $1 \leq j < k$, the terms x_j, x_k are relatively prime.

3. Let $d(n)$ denote the number of divisors of n. Call n a round number if $m < n$ implies $d(m) < d(n)$. Prove that if n is round and

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$$

is the unique prime factorization of n with $p_1 < p_2 < \cdots < p_m$, then there is no prime missing between p_1 and p_m.

4. In a class of 10 students no two of them have the same ordered pair (written and oral examinations) of scores in mathematics. We say that student A is better than B if his two scores are greater than or equal to the corresponding scores of B. The scores are integers between 1 and 5.

(a) Show that there exist three students A, B, and C such that A is better than B and B is better than C.

(b) Would the same be true for a class of 9 students?

5. For a real 2×2 matrix

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

let $\|X\| = x^2 + y^2 + z^2 + t^2$, and define a distance function by $d(X, Y) = \|X - Y\|$. Let $\Sigma = \{X \mid \det(X) = 0\}$ and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find the minimum distance from A to Σ and exhibit a specific matrix $S \in \Sigma$ that achieves this minimum.