2004 Missouri Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the x-axis at a point, say Q. Let $O = (0,0)$ and form the triangle OPQ. Let OQ be the base of this triangle. Find the minimum ratio of the length of the base of $\triangle OPQ$ to its height.

2. The numbers $\pm 1, \pm 2, \ldots \pm 2004$ are written on a blackboard. You decide to pick two numbers x and y at random, erase them, and write their product, xy, on the board. You continue this process until only one number remains. Prove that the last number is positive.

3. A chess position possesses the following property: On every vertical column and on every horizontal row, there is an odd number of pieces. Prove that there is an even number of pieces on black squares.

4. At a point P on the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1,$$

the tangent to the curve meets the x-axis at $(h, 0)$ and the y-axis at $(0, k)$. As P moves on the given curve, find the locus of points $Q(h, k)$.

5. Let a, b, c, and d be integers. Suppose that each of the three quadratics ax^2+bx+c, $ax^2+bx+(c+d)$, and $ax^2+bx+(c+2d)$ factors over the integers, i.e. has rational roots. Let $S = ad > 0$. Show that S represents the area of some Pythagorean triangle (integer-sided right triangle).

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Session II

1. Suppose f is a continuous real-valued function on the interval $[0,1]$. Show that

$$\int_0^1 x^2 f(x)\,dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0,1]$.

2. Prove that $2(3n-1)^n \geq (3n+1)^n$ for all nonnegative integers n.

3. Equations of two ellipses E_1 and E_2 are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2x}{c} = 0 \text{ and } \frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{2x}{c} = 0,$$

respectively. AB is a common tangent, meeting E_1 at A and E_2 at B. Prove that when A and B are joined to the origin O, angle AOB is a right angle.

4. For each positive integer n, let $s(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n

$$s(2n) \leq 2s(n) \leq 10s(2n).$$

5. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, if $n \geq 2$. Use the Fibonacci numbers to express the number K_n of n-tuples $(x_1, x_2, \ldots, x_n)$ of 0's, 1's, and 2's such that 0 is never followed by 1.