2006 Missouri Collegiate Mathematics Competition

Session I

1. Find

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{1}{2n}\right).$$

Solution.

$$\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{1}{2n}$$

$$=\frac{1}{n}\left(\frac{1}{1+\frac{1}{n^2}}+\frac{1}{1+\frac{4}{n^2}}+\frac{1}{1+\frac{9}{n^2}}+\cdots+\frac{1}{1+\frac{n^2}{n^2}}\right)$$

$$=\frac{1}{n}\cdot\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}.$$

This is a Riemann sum. When we take the limit of this as n goes to infinity, the sum changes into an integral given by

$$\int_0^1 \frac{1}{1+x^2}\,dx = \mathrm{Tan}^{-1}x\Big|_0^1 = \frac{\pi}{4}.$$

2. You have a supply of unit squares, 1×2 rectangles and 1×3 rectangles. With these you wish to "tile" a rectangular strip of dimensions $1\times n$ (n is a positive integer). For example, if W_n is the number of ways a $1\times n$ strip can be tiled, then $W_3=4$ since

$$3=2+1=1+2=1+1+1.$$

Determine how many tiling patterns W_n exist when $n=16$ and prove your answer.

Solution.

In tiling a $1\times n$ strip, the *last* piece put in place is either a 1×1 square, a 1×2 rectangle, or a 1×3 rectangle. The 1×1 square comes after W_{n-1} ways of tiling a $1\times(n-1)$ strip. The 1×2 rectangle comes after W_{n-2} ways of tiling a $1\times(n-2)$ strip. The 1×3 rectangle comes after W_{n-3} ways of tiling a $1\times(n-3)$ strip. Hence,

$$W_n = W_{n-1}+W_{n-2}+W_{n-3}$$

for $n\geq 4$. A simple computation then yields $W_{16}=10,609$.

3. Let $\{a_n\}$ be the sequence defined recursively by

$$a_0 = 1,\ a_1 = 0,\ \text{and}\ a_n = \frac{1}{n}a_{n-2} \quad \text{for} \quad n \geq 2.$$

If the function f is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

find the exact value of $f(2)$.

Solution.

Using the recurrence relation it is easy to see that $a_n = 0$ for all odd n. The even terms

$$a_0 = 1,\ a_2 = \frac{1}{2},\ a_4 = \frac{1}{8},\ a_6 = \frac{1}{48},\ \ldots$$

are given by

$$a_{2n} = \frac{1}{2^n n!}.$$

Thus, the function f is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2}\right)^n$$

$$= e^{x^2/2}.$$

So the exact value of $f(2)$ is $f(2) = e^{2^2/2} = e^2$.

4. Find all positive integers c such that $n(n+c)$ is never a perfect square for any positive integer n.

Solution.

Write $n(n+c) = m^2$; then $n^2 + cn - m^2 = 0$, and

$$n = \frac{-c \pm \sqrt{c^2 + 4m^2}}{2}.$$

For n to be integral, it is necessary that $c^2 + 4m^2$ be a perfect square.

Let us consider the situation when $c = 1$, $c = 2$, or $c = 4$ If $c = 1$ or $c = 2$, $c^2 + 4m^2$ is between $(2m)^2$ and $(2m+1)^2$ and so cannot be a perfect. If $c = 4$, then $4^2 + 4m^2 = 4(4 + m^2)$ must be a perfect square.

Therefore, $4+m^2$ must be a perfect square. Since $m^2 < 4+m^2$, $4+m^2 = (m+1)^2$. But then $4 = 2m+1$. This cannot occur since the left and right sides of the equation have opposite parity.

For the remaining values of c we consider 3 cases.

Case 1. $c > 1$ is odd. Choose

$$n = \left(\frac{c-1}{2}\right)^2;$$

then

$$n(n+c) = \left(\frac{c^2-1}{4}\right)^2,$$

so no members in this class are acceptable answers.

Case 2. $c = 2^r,\ r \geq 3$. Choose $n = 2^{r-3}$; then

$$n(n+c) = \left(3 \cdot 2^{r-3}\right)^2,$$

so no members in this class are acceptable answers.

Case 3. $c = 2^r s$, s odd, $s > 1$. Choose

$$n = 2^r \left(\frac{s-1}{2}\right)^2;$$

then

$$n(n+c) = \left(2^r \frac{s^2-1}{4}\right)^2,$$

so no members in this class are acceptable answers.

In summary, the set of integers c requested is $\{1, 2, 4\}$.

5. Let $f(t)$ and $f'(t)$ be differentiable on $[a, x]$ and for each x suppose there is a number c_x such that $a < c_x < x$ and

$$\int_a^x f(t)\,dt = f(c_x)(x-a).$$

Assume that $f'(a) \neq 0$. Then prove that

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

Solution.

Let

$$F(x) = \int_a^x f(t)\,dt.$$

Using Taylor's expansion of $F(x)$, we have

$$F(x) = F(a) + (x-a)F'(a) + \frac{(x-a)^2}{2}F''(\theta_x),$$

where θ_x lies strictly between a and x, and as x goes to a, θ_x also goes to a. We also have $F(a) = 0$, $F'(x) = f(x)$, and $F''(x) = f'(x)$. Thus,

$$F(x) = 0 + (x-a)f(a) + \frac{(x-a)^2}{2}f'(\theta_x).$$

By definition,

$$f(c_x) = \frac{1}{x-a}F(x) = f(a) + \frac{x-a}{2}f'(\theta_x).$$

Therefore,

$$\frac{f(c_x) - f(a)}{x-a} = \frac{1}{2}f'(\theta_x).$$

On the other hand we can write

$$\frac{f(c_x) - f(a)}{x-a}$$

as a product

$$\frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x - a}.$$

On taking the limits of these as x goes to a, we get

$$\lim_{x \to a} \frac{1}{2} f'(\theta_x) = \lim_{x \to a} \frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x - a}.$$

This gives

$$\frac{1}{2} f'(a) = \lim_{x \to a} \frac{f(c_x) - f(a)}{c_x - a} \lim_{x \to a} \frac{c_x - a}{x - a}.$$

In other words,

$$\frac{1}{2} f'(a) = f'(a) \cdot \lim_{x \to a} \frac{c_x - a}{x - a}.$$

This shows

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

2006 Missouri Collegiate Mathematics Competition

Session II

1. The array below is called a magic square because the sum of the three numbers along any row, any column, or the two diagonals, is the same (namely, 15).

8	1	6
3	5	7
4	9	2

(a) Construct a 3×3 multi-magic square, that is, a 3×3 array of 9 distinct integers such that the PRODUCT of the three numbers along any row, any column, or the two diagonals, is the same.

(b) Show that no multi-magic square can be constructed with nine *consecutive* integers.

Solution.

(a) There are many possibilities, but among the easiest to construct might be the following, which makes use of the square above and the property of exponents:

$2^8 = 256$	$2^1 = 2$	$2^6 = 64$
$2^3 = 8$	$2^5 = 32$	$2^7 = 128$
$2^4 = 16$	$2^9 = 512$	$2^2 = 4$

(b) There can be no multi-magic square consisting of nine consecutive integers. Reducing the entries modulo five shows that among nine consecutive integers, there are at most two multiples of 5. Thus, at least one of the columns will have a product which is not a multiple of 5.

2. Evaluate the limit:

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=3}^{n} \binom{k}{3}.$$

Solution.

Using the identity

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

we have

$$\begin{aligned}\sum_{k=3}^{n}\binom{k}{3} &= \binom{3}{3}+\sum_{k=4}^{n}\binom{k}{3}\\ &= \binom{3}{3}+\sum_{k=4}^{n}\left(\binom{k+1}{4}-\binom{k}{4}\right)\\ &= \binom{3}{3}+\binom{n+1}{4}-\binom{4}{4}\\ &= \binom{n+1}{4}=\frac{(n+1)n(n-1)(n-2)}{24}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{(n+1)n(n-1)(n-2)}{24n^4}=\frac{1}{24}.$$

3. Define a sequence of positive integers $\{x_n \mid n = 1, 2, 3, \dots\}$ to be *Dence* if every positive integer can be expressed as a sum of distinct members of the sequence. Now consider the sequence in which $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 7$, $x_5 = 15$, and $x_{k+1} = 2x_k - 7$ for all $k \geq 5$. Prove that this sequence is Dence.

Solution.

By trial all of the positive integers from 1 to $23 (= x_6)$ are expressible as sums of distinct x_k's. Now assume that if $n \geq 6$, all of the integers from 1 to x_n are expressible as sums of distinct x_k's. Choose any $x \in [x_n + 1, x_{n+1}]$. If $x = x_n + 1$, we have trivially $x = x_1 + x_n$. At the other endpoint representation is even more trivial: $x = x_{n+1}$. Finally, if $x \in (x_n + 1, x_{n+1})$, then

$$x_n + 1 + 1 \leq x \leq 2x_n - 7 - 1.$$

Writing $x = x_n + r$, then we have

$$2 \leq r \leq x_n - 8.$$

By assumption,

$$x_n - 8 = \sum_{k<n} x_k$$

holds, so x itself is representable as a sum of distinct x_k's. By the Principle of Mathematical Induction every positive integer is expressible as a sum of distinct x_k's, so $\{x_k \mid k = 1, 2, 3, \dots\}$ is Dence.

Micahel Dorff and Leon Hall, University of Missouri-Rolla

4. For a convex polygon with n sides, let T be a point in the interior of the polygon. Triangulate the polygon by drawing line segments from T to each vertex. Denote the distance from T to side s_i of the polygon by r_i and the area of the corresponding triangle by A_i. Let A be the total area of the polygon. Show that the number r, defined by

$$\frac{1}{r} = \sum_{i=1}^{n} \left(\frac{A_i}{A}\right)\left(\frac{1}{r_i}\right),$$

is independent of the position (inside the polygon) of T.

Solution.

Since

$$A_i = \frac{s_i r_i}{2},$$

we get

$$\frac{A}{r} = \sum_{i=1}^{n} \frac{s_i}{2} = \frac{P}{2},$$

where P is the perimeter of the polygon, making

$$r = \frac{2A}{P}$$

which is independent of the position of T. If the area and perimeter of the polygon are both expressed as functions of r, then

$$\frac{dA}{dr} = P.$$

5. Let

$$\{a_n\}_{n=1}^{\infty} \text{ and } \{\Delta a_n\}_{n=1}^{\infty} = \{a_n - a_{n+1}\}_{n=1}^{\infty}$$

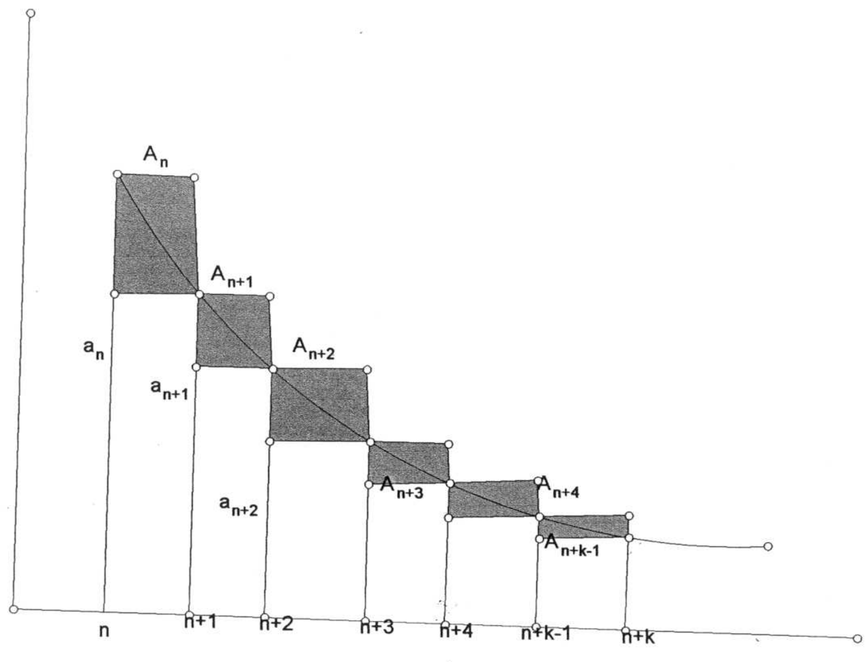
be two decreasing sequences of positive numbers that converge to 0. Prove that the magnitude of the error, $|R_n|$, made in approximating the sum of the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

by its n partial sum is bounded as follows:

$$\frac{a_{n+1}}{2} < |R_n| < \frac{a_n}{2}.$$

Solution.



Consider a function which looks like $f(x) = 1/x$. It should have two properties: (a) $f(n)$ decreases to zero, and (b) $f(n) - f(n+1)$ decreases to zero. Let $f(n) = a_n$.

Note that $a_n = a_n \cdot 1 =$ the area of the rectangle R say, whose base goes from $x = n$ to $x = n+1$, and whose height is a_n. Also, consider the rectangles enclosed by the two ordinates $x = n$, and $x = n+1$, and the horizontal lines $y = a_n$, and $y = a_{n+1}$. Let us call this rectangular area A_n. These areas are shown in the diagram. Also note that for each n, $A_n = a_n - a_{n+1}$.

The important thing to note is that if we translate all the areas A_{n+1}, A_{n+2}, A_{n+3}, A_{n+4}, etc. horizontally to the left, they will fit inside R without overlapping and fill the entire rectangle. In other words,

$$a_n = A_n + A_{n+1} + A_{n+2} + A_{n+3} + A_{n+4} + \cdots .$$

Now consider the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k.$$

This is equal to

$$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + (a_7 - a_8) + \cdots .$$

Let S_n denote its partial sum. Then,

$$S_n = \begin{cases} (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{n-1} - a_n) & \text{if } n \text{ is even,} \\ (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{n-2} - a_{n-1}) + a_n & \text{if } n \text{ is odd.} \end{cases}$$

The error R_n is either

$$R_n = \begin{cases} (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots & \text{if } n \text{ is even,} \\ -a_{n+1} + a_{n+2} - a_{n+3} + a_{n+4} - \cdots & \text{if } n \text{ is odd.} \end{cases}$$

In either case

$$|R_n| = A_{n+1} + A_{n+3} + A_{n+5} + \cdots .$$

We know that

$$a_n = A_n + A_{n+1} + A_{n+2} + A_{n+3} + A_{n+4} + \cdots ,$$

and

$$a_{n+1} = A_{n+1} + A_{n+2} + A_{n+3} + A_{n+4} + \cdots .$$

We will now use the fact that A_n is a decreasing sequence.

$$a_n > A_{n+1} + A_{n+1} + A_{n+3} + A_{n+3} + A_{n+5} + A_{n+5} + \cdots = 2(A_{n+1} + A_{n+3} + A_{n+5} + \cdots) = 2|R_n|.$$

Similarly,

$$a_{n+1} = A_{n+1} + A_{n+2} + A_{n+3} + A_{n+4} + \cdots < A_{n+1} + A_{n+1} + A_{n+3} + A_{n+3} + A_{n+5} + A_{n+5} + \cdots = 2|R_n|.$$

Thus,

$$a_{n+1} < 2|R_n| < a_n.$$