

2007 Missouri Collegiate Mathematics Competition

Session I

1. Seemingly unimportant changes in the terms of a series can have a dramatic effect on the nature of the sum. Thus, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

sums to a transcendental number, but if the  $n^2$  is replaced by  $n^2 + 3n - 4$ , the new series sums to a rational number. Find it.

Solution.

The series is equivalent to

$$\frac{1}{5} \sum_{n=2}^{\infty} \left[ \frac{1}{n-1} - \frac{1}{n+4} \right],$$

for which the  $(5k + 6)$ th partial sum is

$$s_{5k+6} = \frac{1}{5} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) - \frac{1}{5} f_{5k+6},$$

where

$$f_{5k+6} = \frac{1}{5k+6} + \frac{1}{5k+7} + \frac{1}{5k+8} + \frac{1}{5k+9} + \frac{1}{5k+10}.$$

Clearly,

$$0 < f_{5k+6} < \frac{5}{5k+6},$$

so

$$\lim_{k \rightarrow \infty} f_{5k+6} = 0$$

by the Squeeze Theorem. Hence,

$$\lim_{k \rightarrow \infty} s_{5k+6} = \frac{1}{5} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) = \frac{137}{300}.$$

2. A bitwin chain of length one consists of two pairs of twin primes with the property that they are related by being of the form:

$$(n-1, n+1) \text{ and } (2n-1, 2n+1).$$

Prove or disprove that  $30 \mid n$  for  $n \geq 7$ .

Solution.

We will prove that  $30 \mid n$ .

It is easy to see that among any two consecutive numbers (positive integers), one of them is even. Similarly among any three consecutive numbers, one of them is divisible by 3, and also among any 5 consecutive numbers, one of them is divisible by 5. Let us now keep in mind that the numbers  $n - 1$ ,  $n + 1$ ,  $2n - 1$ , and  $2n + 1$  are all primes.

Looking at  $n - 1$  and  $n$ , we see that 2 divides  $n$ .

Looking at  $n - 1$ ,  $n$ , and  $n + 1$ , we note that 3 must divide  $n$ .

Looking at the five numbers,  $2(n - 1)$ ,  $2n - 1$ ,  $2n$ ,  $2n + 1$ , and  $2(n + 1)$ , we note that 5 must divide  $n$ .

Hence, 30 divides  $n$ .

The first few values of  $n$  generating bitwin chains of length one are 6, 30, 660, 810, 2130, 2550, 3330, . . . (Sloane's A066388). For more on bitwin chains of length  $i \geq 1$ , see <http://mathworld.wolfram.com/BitwinChain.html>

3. Let  $f$  be the complex function

$$f(z) = e^{z^2}.$$

Find the set of all points  $z$  in the complex plane for which  $f(z)$  is a real number. Give a geometric description of this set.

Solution.

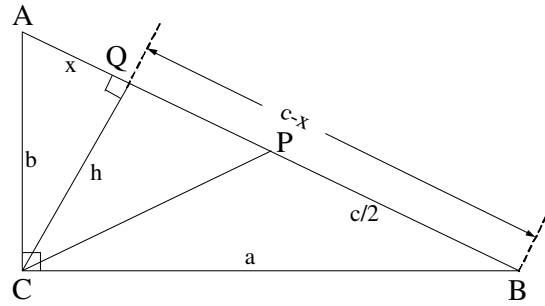
Write  $z$  as  $x + iy$ . Then

$$\begin{aligned} f(z) &= e^{z^2} = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2} e^{2ixy} = e^{x^2-y^2} (\cos 2xy + i \sin 2xy) \\ &= (\cos 2xy)e^{x^2-y^2} + i(\sin 2xy)e^{x^2-y^2}. \end{aligned}$$

So  $\text{Im}(f(z)) = 0$  if and only if  $\sin(2xy)e^{x^2-y^2} = 0$ . Since  $e^{x^2-y^2}$  can't be zero, we have to have  $\sin(2xy) = 0$ . Therefore,  $f(z)$  will be a real number if  $2xy = k\pi$  for any integer  $k$ . Thus, the set of all points  $z$  in the complex plane for which  $f(z)$  is a real number will be the  $x$ - and  $y$ -axes and all of the hyperbolas  $xy = (k\pi)/2$ , where  $k$  is any integer.

4. A triangle is Pythagorean if it is a right triangle and the lengths of all of its sides are integers. Suppose that  $\triangle ABC$  is Pythagorean; for concreteness assume that the lengths of the three sides satisfy  $c > a > b$ . The median and the altitude are now drawn from  $C$  to the hypotenuse, where they meet the latter at  $P$ ,  $Q$ , respectively. Determine simple conditions upon  $a$ ,  $b$ ,  $c$  so that  $\triangle CQP$  will also be Pythagorean.

Solution.



We have

$$b^2 - x^2 = a^2 - (c - x)^2,$$

from which

$$x = \frac{b^2 - a^2 + c^2}{2c} = \frac{b^2}{c}.$$

Then

$$h = \overline{CQ} = \sqrt{b^2 - x^2} = \sqrt{b^2 - \frac{b^4}{c^2}} = \frac{ab}{c},$$

upon making use of  $c^2 - b^2 = a^2$ .

Next,

$$\overline{PQ} = c - x - \frac{c}{2} = \frac{c}{2} - \frac{b^2}{c} = \frac{a^2 - b^2}{2c}.$$

Finally,

$$\overline{CP} = \sqrt{h^2 - \overline{PQ}^2} = \frac{a^2 + b^2}{2c} = \frac{c}{2}.$$

Summarizing:

1.  $\overline{CP}$  is integral if  $c$  is even;
2.  $\overline{PQ}$  is integral if  $c$  divides  $b^2$ ;
3.  $\overline{CQ}$  is integral if  $c$  divides  $ab$ .

Equivalent formulations are possible.

5. Associated with the Fibonacci numbers  $\{F_n\}_{n=1}^{\infty}$ , are the Fibonacci polynomials,  $\{U_n(x)\}_{n=1}^{\infty}$ ,

$$U_n(x) = \begin{cases} 1, & n = 1; \\ x, & n = 2; \\ xU_{n-1}(x) + U_{n-2}(x), & n > 2. \end{cases}$$

- (a) Find a function  $F(x, y)$  such that (formally)

$$F(x, y) = \sum_{n=1}^{\infty} U_n(x)y^n.$$

Such a function is called a generating function for the  $U_n(x)$ 's.

(b) Establish that for each  $n$ ,  $U_n(1) = F_n$ .

Solution.

(a) We have

$$F(x, y) - U_1(x)y - U_2(x)y^2 = \sum_{n=3}^{\infty} U_n(x)y^n \quad (1)$$

$$xyF(x, y) - xU_1(x)y^2 = \sum_{n=3}^{\infty} xU_{n-1}(x)y^n \quad (2)$$

$$y^2F(x, y) = \sum_{n=3}^{\infty} U_{n-2}(x)y^n, \quad (3)$$

upon multiplication of  $F(x, y)$  by one and two powers of  $y$ . Subtraction of equations (2,3) from equation (1) and substitution of  $U_1(x)$  by 1 and  $U_2(x)$  by  $x$  yield

$$\begin{aligned} & F(x, y) - y - xy^2 - xyF(x, y) + xy^2 - y^2F(x, y) \\ &= \sum_{n=3}^{\infty} \left( U_n(x) - xU_{n-1}(x) - U_{n-2}(x) \right) y^n. \end{aligned}$$

The right-hand side is identically 0, so solving for  $F(x, y)$ , we obtain

$$F(x, y) = \frac{y}{1 - xy - y^2}.$$

(b) Quick Solution:  $x = 1$  implies

$$F(1, y) = \sum_{n=1}^{\infty} U_n(1)y^n = \frac{y}{1 - y - y^2},$$

which is the generating function  $G(y)$  for the Fibonacci numbers themselves, that is,

$$G(y) = \frac{y}{1 - y - y^2} = \sum_{n=1}^{\infty} F_n y^n.$$

Alternatively, one can show that  $U_n(1) = F_n$  for each  $n$  by means of Mathematical Induction.

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Session II

1. Suppose that

$$\frac{2x + 3}{x^2 - 2x + 2}$$

has the Taylor series

$$\sum_{k=0}^{\infty} a_k x^k.$$

Find

$$\sum_{k=0}^{\infty} a_k.$$

Solution.

Let

$$F(x) = \frac{2x + 3}{x^2 - 2x + 2}.$$

To use the Taylor series, it is necessary to determine the circle of convergence. Note that the roots of the denominator are  $1 \pm i$ , so the radius of convergence for the Taylor series will be  $\sqrt{2}$ . Therefore,

$$\frac{2x + 3}{x^2 - 2x + 2} = \sum_{k=0}^{\infty} a_k x^k$$

for all  $x$  which satisfy  $|x| < \sqrt{2}$ . This gives us

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_k (1)^k = \frac{2(1) + 3}{1^2 - 2(1) + 2} = 5.$$

2. Find the point on a given line such that the sum of its distances from two fixed points is a minimum. Assume the two fixed points and the given line are in the same plane.

Solution.

If one of the two points is on the line, then the solution is clearly that point. It is also clear that if the two points are on opposite sides of the line, then the solution is the intersection of the line with the segment connecting the points. Now assume the two points are on the same side of the line. Because we are interested in the sum of distances from two fixed points, consider the family of ellipses with foci at the two points, and denote the foci  $P_1$  and  $P_2$ . If an ellipse in this family intersects the line in two points, say  $A$  and  $B$ , then from the definition of an ellipse, the sum of the distances from  $A$  to the foci equals the sum

of the distances from  $B$  to the foci. If a smaller ellipse in the family also intersects the line in two points, the same thing happens, except that the sum of the distances is less for the smaller ellipse. Therefore, if we continue to shrink the ellipse until it is tangent to the line, the point of tangency will be where the required minimum occurs.

This is basically the “find the shortest distance from the house to the river to the barn” problem found in most calculus books in the Lagrange multiplier section, but as the above analysis shows, a non-calculus solution is possible. (see: Note on Problem 118, by George R. Dean, *American Mathematical Monthly*, Volume 7, Number 3 (March 1900), p. 78.)

3. Assume the point  $(h, k)$  lies “outside” the circle  $x^2 + y^2 = r^2$ ,  $0 < r < \sqrt{h^2 + k^2}$ . Find an expression for the  $x$ -intercepts of the tangent lines to the circle from the point  $(h, k)$  in terms of  $h$ ,  $k$ , and  $r$ .

Solution.

Represent both intercepts as  $(a, 0)$  so that the slope of the tangent lines from  $(h, k)$  will be written as  $m = k/(h - a)$ . By implicit differentiation the slope of the tangent lines to the circle is given as  $m = -x/y$ . The points of intersection of the two tangent lines from  $(h, k)$  with the circle will satisfy the equation

$$y = -\frac{x}{y}(x - a), \text{ or } x^2 + y^2 = ax;$$

thus

$$ax = r^2, \text{ or } x = \frac{r^2}{a}, \text{ and } y = \pm \frac{r}{a} \sqrt{a^2 - r^2}$$

(the sign will depend on the location of the point  $(h, k)$ ). Equating the two formulas for the slope gives

$$\frac{k}{h - a} = -\frac{x}{y} = -\frac{\frac{r^2}{a}}{\pm \frac{r}{a} \sqrt{a^2 - r^2}} = m \frac{r}{\sqrt{a^2 - r^2}}$$

so that squaring both sides and cross-multiplying produces the equation

$$k^2(a^2 - r^2) = r^2(h - a)^2.$$

Expanding and combining terms generates the quadratic equation

$$(k^2 - r^2)a^2 + 2hr^2a - r^2(h^2 + k^2) = 0$$

which when solved for  $a$  gives two values

$$a = \frac{-hr^2 \pm kr\sqrt{h^2 + k^2 - r^2}}{k^2 - r^2}.$$

4. Let

$$S = \{5a + 503b : a \text{ and } b \text{ are nonnegative integers}\}.$$

What is the largest integer which does NOT belong to  $S$ ?

Solution.

Let  $x$  be any positive integer. According to the division algorithm,  $x = 5k + r$ , where  $r$  is 0, 1, 2, 3, or 4. We consider the following five cases.

Case 1:  $x = 5k$  must lie in the set  $S$ .

Case 2:  $x = 5k + 1 = 5(k - 201) + 503(2)$  will lie in the set  $S$  so long as  $k - 201 \geq 0$ .

Case 3:  $x = 5k + 2 = 5(k - 402) + 503(4)$  will lie in the set  $S$  so long as  $k - 402 \geq 0$ .

Case 4:  $x = 5k + 3 = 5(k - 100) + 503(1)$  will lie in the set  $S$  so long as  $k - 100 \geq 0$ .

Case 5:  $x = 5k + 4 = 5(k - 301) + 503(3)$  will lie in the set  $S$  so long as  $k - 301 \geq 0$ .

In all five cases, we are able to guarantee that  $x$  lies in  $S$  as long as the value of  $k$  is large enough. The largest value of  $k$  for which we can't guarantee that  $x$  lies in  $S$  is  $k = 401$  (which occurred in Case 3). In this case,  $x = 5(401) + 2 = 2007$ . It is easy to show that 2007 does not belong to  $S$  by considering the equation  $5a + 503b = 2007$  over the nonnegative integers  $a$  and  $b$ :

Reducing  $5a + 503b = 2007$  modulo 5 yields

$$5a + 503b \equiv 2007 \pmod{5}$$

$$503b \equiv 2007 \pmod{5}$$

$$3b \equiv 2 \pmod{5}$$

$$b \equiv 4 \pmod{5}.$$

The smallest nonnegative integer  $b$  satisfying this congruence is  $b = 4$ , which forces  $a$  to be negative. Thus, 2007 does not lie in  $S$ .

Note: This is a special case of Problem 247 in the March 1983 edition of the *College Mathematics Journal*, which was submitted by Mangho Ahuja.

5. Does there exist a real valued continuous function  $f$  with domain the set of all real numbers (usual topology on the domain and range) such that if  $x$  is rational then  $f(x)$  is irrational and if  $x$  is irrational then  $f(x)$  is rational? Prove your answer.

Solution.

We will show by contradiction that no such function  $f$  exists.

Suppose that such a function  $f$  does exist and consider the function  $h(x) = f(x) + x$ . Note that  $h$  is a continuous function. Moreover for every value of  $x$ ,  $h(x)$  is irrational. Thus, the range of  $h$  is a subset of the irrationals. Since  $h$  is continuous, its range must be a connected subset of the real numbers. The only connected subsets of the real numbers containing only irrational numbers are singletons  $\{k\}$ . Therefore, the range of  $h$  is a single (irrational) value  $k$ . This implies that  $f(x) = k - x$  for all  $x$ . However, this gives us a contradiction, since  $-k$  and  $2k$  are irrational and  $f(-k) = 2k$ .