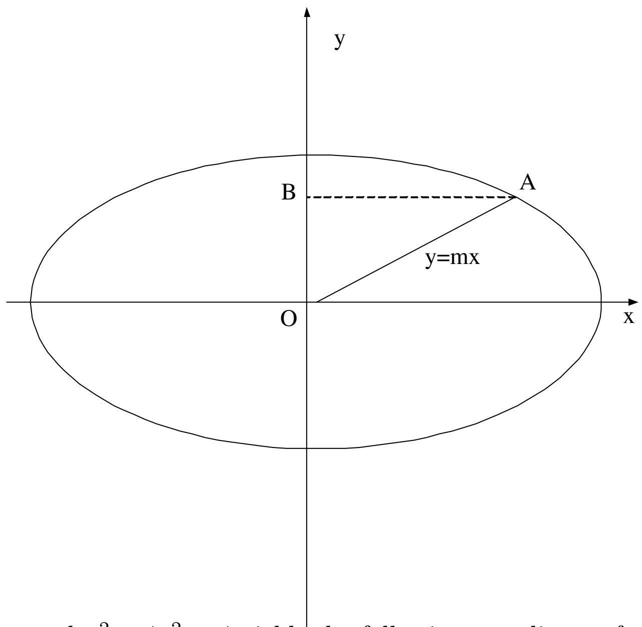
2008 Missouri Collegiate Mathematics Competition

Session I

1. A straight line segment of arbitrary positive slope m is drawn from the origin O to the point of intersection A in quadrant I with the ellipse $x^2 + 4y^2 = 4$. A line segment is then drawn parallel to the x-axis from A over to the y-axis, which the segment meets at point B. Points O, A, B are thus the vertices of a right triangle. Deduce the value of m that maximizes the area R of triangle OAB, and prove that this really is the maximum area.

Solution.



Combination of $y = mx$ and $x^2 + 4y^2 = 4$ yields the following coordinates for point A:

$$\left(\frac{2}{\sqrt{1+4m^2}}, \frac{2m}{\sqrt{1+4m^2}} \right).$$

Hence, the area of the triangle is

$$R = \frac{1}{2}(\overline{AB})(\overline{OB}) = \frac{1}{2} \frac{2}{\sqrt{1+4m^2}} \frac{2m}{\sqrt{1+4m^2}} = \frac{2m}{1+4m^2}.$$

Then

$$\frac{dR}{dm} = \frac{2-8m^2}{(1+4m^2)^2} = 0,$$

the only solution of which that is consistent with $m > 0$, is $m = 1/2$. To this corresponds

$$R = \frac{2(1/2)}{1+4(1/2)^2} = \frac{1}{2}.$$

That this is the maximum area we can see from the computation

$$\begin{aligned}\frac{d^2R}{dm^2} &= \frac{(1+4m^2)^2(-16m) - (2-8m^2)(2)(1+4m^2)(8m)}{(1+4m^2)^4} \\ &= \frac{16m}{(1+4m^2)^3}(4m^2-3),\end{aligned}$$

the value of which is -2 at $m = 1/2$.

2. Let a be a positive real number. The Lemniscate of Bernoulli is defined by

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

Find the area bounded by the Lemniscate of Bernoulli.

Solution.

In polar coordinates, the Lemniscate of Bernoulli is defined by

$$r^2 = a^2\cos 2\theta.$$

Therefore, the area bounded by the Lemniscate of Bernoulli is

$$2\int \frac{1}{2}r^2\,d\theta = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\,d\theta = a^2.$$

3. The sequence of Catalan numbers, $\{C_n\}_{n=1}^{\infty}$, is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Does there exist a member of the sequence that is not a natural number? Find one, or prove that there is none.

Solution.

The idea is to rewrite each C_n as a linear combination of binomial coefficients. We have

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n}\frac{(2n)!}{(n+1)!(n-1)!}$$

so

$$\begin{cases} (n+1)C_n = \binom{2n}{n} \\ nC_n = \binom{2n}{n-1}. \end{cases}$$

Subtraction gives

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

But the two binomial coefficients are natural numbers because they are numbers of combinations of countable objects. Additionally,

$$\binom{2n}{n} > \binom{2n}{n-1}$$

if and only if

$$\frac{1}{n} > \frac{1}{n+1},$$

which is true for all $n \geq 1$. Hence, C_n is the positive difference of two natural numbers for all n, and so is itself always a natural number.

4. Suppose a belt is stretched tightly over two circular pulleys with radii r_1 and r_2, whose centers are d units apart with $d > r_1 + r_2$. If $r_1 > r_2$, find a formula for the total length of the belt in terms of r_1, r_2, and d.

Solution.

The total length is composed of two equal straight parts and two circular arcs. View the pulleys as side-by-side horizontally with the larger one on the left. Denote the centers of the pulleys by P_1 (larger pulley) and P_2 and the top points of tangency by T_1 (on larger pulley) and T_2. Let θ be the angle $T_1P_1P_2$. Then we get:

$$L = 2\left[\sqrt{d^2 - (r_1 - r_2)^2} + r_1(\pi - \theta) + r_2\theta\right],$$

where

$$\theta = \cos^{-1}\left(\frac{r_1 - r_2}{d}\right)$$

is in radians.

5. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}}$$

for $n \geq 2$. What is this series when $n = 1$?

Solution.

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}} = \sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{n}} = \sum_{k=1}^{\infty} \frac{1}{\frac{(k+1)(k+2)\cdots(k+n)}{n!}}$$

$$= n! \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+n)}$$

$$= \frac{n!}{n-1} \sum_{k=1}^{\infty} \left(\frac{1}{(k+1)(k+2)\cdots(k+n-1)} - \frac{1}{(k+2)(k+3)\cdots(k+n)} \right)$$

$$= \frac{n!}{n-1} \left(\frac{1}{2 \cdot 3 \cdot \cdots \cdot n} - \frac{1}{3 \cdot 4 \cdot \cdots \cdot (n+1)} + \frac{1}{3 \cdot 4 \cdot \cdots \cdot (n+1)} - \frac{1}{4 \cdot 5 \cdot \cdots \cdot (n+2)} \right.$$

$$\left. + \frac{1}{4 \cdot 5 \cdot \cdots \cdot (n+2)} - \frac{1}{5 \cdot 6 \cdot \cdots \cdot (n+3)} + \cdots \right)$$

$$= \frac{n!}{n-1} \cdot \frac{1}{n!} = \frac{1}{n-1}.$$

For $n = 1$, the series is a harmonic series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which is divergent, and the formula $1/(n-1)$ would indicate that the series should be divergent.

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Session II

1.

(a) Let $f(x) = x^3 + x$. Let $g(x)$ be the inverse function of $f(x)$. Find $g'(10)$.

(b) For $x > 0$, define $h(x) = 1/f(x)$. Prove that the function $f(x) + h(x)$ has its absolute minimum when $x = g(1)$.

Solution.

(a) Since f and g are inverse functions, $f(g(x)) = x$. (The equation $g(f(x)) = x$ will work as well.) On differentiating we get

$$f'(g(x)) \cdot g'(x) = 1.$$

Since $f(2) = 10$, we have $g(10) = 2$. Substituting $x = 10$ in the previous equation,

$$f'(g(10)) \cdot g'(10) = 1, \quad \text{or} \quad f'(2) \cdot g'(10) = 1.$$

But $f'(x) = 3x^2 + 1$, so $f'(2) = 13$. Hence, $g'(10) = 1/13$.

(b) Let $F(x) = f(x) + h(x) = f(x) + 1/f(x)$. Then

$$F'(x) = f'(x) - \frac{f'(x)}{[f(x)]^2} = 0, \quad \text{so} \quad f'(x)\{[f(x)]^2 - 1\} = 0.$$

As $f'(x) > 1$, then division by $f'(x)$ is permitted, and $f(x_0) = 1$ (since $f(x) > 0$) at the critical point x_0. Thus, $g(f(x_0)) = g(1)$. To determine the nature of the critical point, we compute

$$\begin{aligned} F''(x) &= f''(x) - \frac{[f(x)]^2 f''(x) - 2f(x)[f'(x)]^2}{[f(x)]^2} \\ &= 6g(1) - \frac{1^2[6g(1)] - 2(1)[3(g(1))^2 + 1]^2}{1^2} \\ &= 2[3(g(1))^2 + 1]^2 \quad \text{at } x = x_0 \\ &> 0, \quad \text{necessarily.} \end{aligned}$$

Hence, $x = x_0$ is the location of a relative minimum. It is the location of the absolute minimum because at the endpoints

$$\lim_{x \to 0} F(x) = \lim_{x \to \infty} F(x) = \infty.$$

Note: We do not need to compute x_0 explicitly; for completeness, it is roughly $x_0 = 0.682328$.

2. Consider the lattice $L = \{(x, y) \ : \ x, y \in \mathbb{Z}\}$. Color the lattice using an arbitrary coloring scheme with 2008 available colors. Prove or disprove: In every coloring scheme, there must be a rectangle whose four vertices all lie in L and are colored with the same color.

Solution.

Such a rectangle must exist. First, note that, by the Pigeonhole Principle, among any set of 2009 points from L, at least two must have the same color. In particular, for a fixed integer x, at least two of the points $(x, 0)$, $(x, 1)$, $(x, 2) \ldots$, $(x, 2008)$, must have the same color. For these 2009 points, there are 2008^{2009} different possible colorings. Therefore, by the Pigeonhole Principle (again), as x runs from 0 to 2008^{2009}, there must be at least two values of x (say x_1 and x_2) for which the coloring for

$$(x_1, 0),\ (x_1, 1),\ (x_1, 2),\ \ldots,\ (x_1, 2008) \text{ and } (x_2, 0),\ (x_2, 1),\ (x_2, 2),\ \ldots,\ (x_2, 2008)$$

are the same. (By the same, we mean that the corresponding points (x_1, y) and (x_2, y) are the same color for each $y = 0, 1, 2, \ldots, 2008$.) And, as noted before, among the points

$$(x_1, 0),\ (x_1, 1),\ (x_1, 2),\ \ldots,\ (x_1, 2008),$$

there must be two y-values (say y_1 and y_2) for which (x_1, y_1) and (x_1, y_2) have the same color. Thus, the points (x_1, y_1), (x_1, y_2), (x_2, y_1), and (x_2, y_2) have the same color, and are the vertices for some rectangle.

Exercise 1.6 (p. 17) from *Ramsey Theory on the Integers,* by Bruce M. Landman and Aaron Robertson, AMS publication.

3. Find $b > 1$ such that the graphs of $y = \log_b(x)$ and $y = b^x$ intersect in exactly one point, i.e., are tangent to one another.

Solution.

Because $y = \log_b(x)$ and $y = b^x$ are inverse functions, any intersections must occur on the line $y = x$, and for the case when the graphs are tangent, the common tangent line must be $y = x$ as well. At the point of tangency, say x_0, then, the derivative of each function must be one, giving:

$$(b^x)'_{x=x_0} = b^{x_0} \ln b = 1$$

$$(\log_b x)'_{x=x_0} = \frac{1}{x_0 \ln b} = 1.$$

From the second equation we get $x_0 = 1/(\ln b)$, and substituting this into the first equation gives

$$\frac{1}{\ln b} = \log_b\left(\frac{1}{\ln b}\right) = \frac{\ln\left(\frac{1}{\ln b}\right)}{\ln b}$$

which leads to

$$\ln\left(\frac{1}{\ln b}\right) = 1,$$

which then gives

$$b = e^{\frac{1}{e}} = \exp\left(\frac{1}{e}\right).$$

(Substituting back to get x_0 yields $x_0 = e$.)

4. Suppose that

$$\frac{2x+3}{x^2 - 2x + 2}$$

has the Taylor series

$$\sum_{k=0}^{\infty} a_k x^k.$$

Find the sum of the odd numbered coefficients, i.e., find

$$\sum_{k=0}^{\infty} a_{2k+1} = a_1 + a_3 + a_5 + \cdots .$$

Solution.

Let

$$F(x) = \frac{2x+3}{x^2-2x+2}.$$

If the Taylor series for F is to represent F, then the series must converge inside the circle in the complex plane that is centered at the origin and extends out to the nearest point at which the denominator of F vanishes. This occurs when $x = 1 \pm i$, so the radius R of the circle of convergence is

$$R = \{(1 \pm i)(1 \mp i)\}^{1/2} = \sqrt{2}.$$

Therefore,

$$\frac{2x+3}{x^2-2x+2} = \sum_{k=0}^{\infty} a_k x^k$$

for all which satisfy $|x| < \sqrt{2}$. This gives us

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_k (1)^k = F(1) \quad \text{and} \quad \sum_{k=0}^{\infty} (-1)^k a_k = \sum_{k=0}^{\infty} a_k (-1)^k = F(-1).$$

Thus,

$$\sum_{k=0}^{\infty} a_{2k+1} = \frac{1}{2}\Big((a_0 + a_1 + a_2 + a_3 + \cdots) - (a_0 - a_1 + a_2 - a_3 + \cdots)\Big)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} (-1)^k a_k\right) = \frac{1}{2}\Big(F(1) - F(-1)\Big)$$

$$= \frac{5 - \frac{1}{5}}{2} = \frac{12}{5}.$$

5. For each integer n, let $a_n = 8n^2 + 3n + 10$ and $b_n = 3n^2 + n + 3$. Since $a_1 = 21$ and $b_1 = 7$, we can write $\gcd(a_1, b_1) = 7$, where gcd denotes the greatest common divisor. Find $\max_{n \in \mathbb{Z}} \gcd(a_n, b_n)$.

Solution.

Suppose that d is a common divisor of a_n and b_n. Then d must also divide

$$3a_n - 8b_n = n + 6.$$

Dividing a_n and b_n by $n + 6$, we get

$$8n^2 + 3n + 10 = (n + 6)(8n - 45) + 280 \quad \text{and} \quad 3n^2 + n + 3 = (n + 6)(3n - 17) + 105.$$

Therefore, d must divide both 280 and 105. Since the greatest common divisor of 280 and 105 is 35, this is our (potential) answer. However, we need to check that this gcd is actually attained. To do this, we simply use $n = -6$:

$$a_{-6} = 8(-6)^2 + 3(-6) + 10 = 280 \quad \text{and} \quad b_{-6} = 3(-6)^2 + (-6) + 3 = 105.$$