2009 Missouri Collegiate Mathematics Competition

Session I

1. For the parabola having equation $y = -x^2$ let $a < 0$ and $b > 0$ with $P : (a, -a^2)$ and $Q : (b, -b^2)$. Let M be the midpoint of PQ and let R be the intersection of the vertical line through M with the parabola. Finally, let l be the tangent line to the parabola at Q. Prove that every vertical line segment with one end on PQ and the other end on l is bisected by the line through Q and R.

2. Prove that for any x in the half-open interval $(0, \pi/2]$ one has

$$\left(\frac{\sin x}{x}\right)^3 > \cos x.$$

3. Let A be a set with $|A| = n$, and let k be a positive integer. Determine the number of subset sequences of the form $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq A$.

4. Find the value of the infinite product

$$\left(\frac{7}{9}\right) \cdot \left(\frac{26}{28}\right) \cdot \left(\frac{63}{65}\right) \cdot \cdots = \lim_{n\to\infty} \prod_{k=2}^{n} \left(\frac{k^3 - 1}{k^3 + 1}\right).$$

5. Evaluate

$$\iiint_S \min\{x, y, z\}\, \mathrm{d}V,$$

where $S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.

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Session II

1. A piece of wire of length x is cut into two pieces, one of which is formed into a square and the other into a circle, such that the total area enclosed by the two figures is a positive constant A. Find the ratio of the length of the edge of the square to the length of the radius of the circle that makes the length of the wire a maximum. Similarly, find the ratio that makes the length of the wire a minimum.

2. Determine the number of subsets S of the set $\{1, 2, \ldots, n\}$ such that S contains no two consecutive integers. Express the answer in terms of the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$) and prove your answer.

3. Consider a piece of paper glued to the outside of the cylinder $x^2 + y^2 = 1$. Suppose that we open a compass to a radius r ($0 < r < 2$), put the stationary point of the compass at the point $(1, 0, 0)$ on the cylinder, and draw a "circle" on the paper (that is, we use the pencil end of the compass to draw a curve).

If we now remove the paper from the cylinder and draw a coordinate system with the origin at the stationary compass point, the Y axis in the same direction as the original z axis, and the X axis oriented appropriately, we can now consider the "circle" as a plane figure. Find an equation for this figure in the XY-plane.

4. Determine

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin\left(\frac{\pi/2}{k}\right) - \cos\left(\frac{\pi/2}{k}\right) - \sin\left(\frac{\pi/2}{k+2}\right) + \cos\left(\frac{\pi/2}{k+2}\right) \right).$$

5. Mersenne primes continue to make news. A number $M_p = 2^p - 1$ is a Mersenne prime if and only if p is prime and $2^p - 1$ is also prime. Let the operator DS denote "form the sum of the digits"; for example, $DS(5119) = 16$. Let the operator DR denote "execute DS repeatedly until a result in the interval $[1, 9]$ is obtained"; for example, $DR(5119) = DS^2(5119) = DS(16) = 7$.

(a) Prove the following lemma.

<u>Lemma</u>. If A and B are natural numbers, then

$$DR(AB) = DR(DR(A) \cdot DR(B)).$$

(b) Prove that for any Mersenne prime greater than 7, $DR(M_p) = 1$ or 4. (You may use the Lemma in part (a) without proving part (a)).