2009 Missouri Collegiate Mathematics Competition

Session I

1. For the parabola having equation $y = -x^2$ let $a < 0$ and $b > 0$ with $P : (a, -a^2)$ and $Q : (b, -b^2)$. Let M be the midpoint of PQ and let R be the intersection of the vertical line through M with the parabola. Finally, let l be the tangent line to the parabola at Q. Prove that every vertical line segment with one end on PQ and the other end on l is bisected by the line through Q and R.

Solution.

The equation of line l is

$$y = -2bx + b^2.$$

The equation of the line containing P and Q is

$$y = -(b + a)x + ab.$$

The equation of the line containing Q and R is

$$y = -\left(\frac{a + 3b}{2}\right) x + \left(\frac{ab + b^2}{2}\right).$$

The y-coordinate of the midpoint of the vertical segment from PQ to l is the average ordinate of the first two lines:

$$y_{\text{avg}} = \frac{1}{2}(-2bx + b^2 - (b + a)x + ab)$$

and the right side simplifies to the right side of the equation of the line through Q and R.

Remark. This problem is a part of the method Archimedes used in his discovery that the area of a segment of a parabola (in this problem, the area between the parabola and line segment PQ) is equal to $\frac{4}{3}$ of the area of the triangle formed by P, Q, and the vertex of the segment of the parabola between P and Q (in this problem, the point R). Note that the vertex of a segment of a parabola is not necessarily the same as the vertex of the whole parabola.

2. Prove that for any x in the half-open interval $(0, \pi/2]$ one has

$$\left(\frac{\sin x}{x}\right)^3 > \cos x.$$

Solution.

On $(0, \pi/2]$,

$$\begin{cases} \sin x > x - \frac{x^3}{6} \\ \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \end{cases}$$

so

$$\left(\frac{\sin x}{x}\right)^3 > \cos x \quad \text{if} \quad \left(1-\frac{x^2}{6}\right)^3 > 1-\frac{x^2}{2}+\frac{x^4}{24}$$

$$\text{iff} \quad \frac{x^4}{12}-\frac{x^6}{216} > \frac{x^4}{24}$$

$$\text{iff} \quad \frac{x^4}{24} > \frac{x^6}{216}$$

$$\text{iff} \quad 3 > x.$$

But $x \leq \pi/2 \approx 1.57 < 3$, so the desired result follows.

3. Let A be a set with $|A| = n$, and let k be a positive integer. Determine the number of subset sequences of the form $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq A$.

Solution.

For any $a \in A$, either $a \notin S_k$ or there is the least i for which $a \in S_i$ where $1 \leq i \leq k$. Therefore, there are $k+1$ such choices for each a. We conclude that there are $(k+1)^n$ ways to form sequence $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq A$.

4. Find the value of the infinite product

$$\left(\frac{7}{9}\right)\cdot\left(\frac{26}{28}\right)\cdot\left(\frac{63}{65}\right)\cdot\cdots = \lim_{n\to\infty}\prod_{k=2}^{n}\left(\frac{k^3-1}{k^3+1}\right).$$

Solution.

We rewrite the nth partial product so as to reveal two sets of telescoping products:

$$\begin{aligned}
\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} &= \prod_{k=2}^{n} \left(\frac{k-1}{k+1}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{k^2-k+1}\right) \\
&= \prod_{k=2}^{n} \left(\frac{k-1}{k+1}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right) \\
&= \prod_{k=2}^{n} \left(\frac{(k-1)((k-1)+1)}{k(k+1)}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right) \\
&= \frac{2}{n(n+1)} \cdot \frac{n^2+n+1}{3} \\
&= \frac{2}{3}\left(1+\frac{1}{n(n+1)}\right).
\end{aligned}$$

Hence,

$$\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1} = \frac{2}{3} \lim_{n\to\infty} \left(1+\frac{1}{n(n+1)}\right) = \frac{2}{3}.$$

<u>Remark</u>. The problem was suggested by Nick Hobson.

5. Evaluate

$$\iiint_S \min\{x,y,z\}\,\mathrm{d}V,$$

where $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.

Solution.

Inside the square-based pyramid P bounded by $z=0$, $x=1$, $y=1$, $z=y$, and $z=x$ either the inequality $z \le x \le y$ or $z \le y \le x$ holds. In either case $\min\{x,y,z\} = z$ in this region, and

$$\iiint_P \min\{x,y,z\}\,\mathrm{d}V = \int_0^1 \int_z^1 \int_z^1 z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{12}.$$

By symmetry, there are two more such pyramids, one where the minimum coordinate is x and another where the minimum coordinate is y. Thus,

$$\iiint_S \min\{x,y,z\}\,\mathrm{d}V = 3\left(\frac{1}{12}\right) = \frac{1}{4}.$$

Or, consider the triangle-based pyramid in which $z \le y \le x$ (half of the one above), bounded by $z = 0$, $x = 1$, $z = y$, and $y = x$. Integrating over this pyramid, we get

$$\int_0^1 \int_0^x \int_0^y z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{24}.$$

There are six such pyramids, making the desired integral, as before, 1/4.

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Session II

1. A piece of wire of length x is cut into two pieces, one of which is formed into a square and the other into a circle, such that the total area enclosed by the two figures is a positive constant A. Find the ratio of the length of the edge of the square to the length of the radius of the circle that makes the length of the wire a maximum. Similarly, find the ratio that makes the length of the wire a minimum.

Solution.

Let s be the length of an edge of the square and r be the length of the radius of the circle. Then

$$x = 4s + 2\pi r \quad \text{and} \quad A = s^2 + \pi r^2.$$

Let $t = s/r$, the ratio of the edge of the square to the radius of the circle. Calculating r and s in terms of t gives

$$r = \sqrt{\frac{A}{t^2+\pi}} \quad \text{and} \quad s = t\sqrt{\frac{A}{t^2+\pi}}.$$

Then

$$x = 4t\sqrt{\frac{A}{t^2+\pi}} + 2\pi\sqrt{\frac{A}{t^2+\pi}} = (4t+2\pi)\sqrt{\frac{A}{t^2+\pi}}, \quad t \in [0,\infty).$$

Differentiating x with respect to t gives

$$x' = \frac{2\sqrt{A}}{(t^2+\pi)^{3/2}}(-\pi t + 2\pi),$$

so $t = 2$ is the critical value in the domain. $x' > 0$ for $t < 2$ and $x' < 0$ for $t > 2$. At $t = 0$, $x = 2\sqrt{\pi A}$; at $t = 2$, $x = 2\sqrt{A(4+\pi)}$; and as $t \to \infty$, $x \to 4\sqrt{A}$. The maximum thus occurs at $t = 2$, and the minimum occurs at $t = 0$.

2. Determine the number of subsets S of the set $\{1, 2, \ldots, n\}$ such that S contains no two consecutive integers. Express the answer in terms of the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$) and prove your answer.

Solution.

Let K_n be the number of such subsets S. If $n \in S$, then $n-1 \notin S$. The rest of the elements of S are from $\{1, 2, \ldots, n-2\}$. Thus, there are K_{n-2} such S. If $n \notin S$, then S consists of elements of $\{1, 2, \ldots, n-1\}$ and there are K_{n-1} such subsets S. Thus, $K_n = K_{n-1} + K_{n-2}$ for $n \geq 3$, with $K_1 = 2$ and $K_2 = 3$. Therefore, $K_n = F_{n+2}$ since $F_3 = 2$ and $F_4 = 3$.

3. Consider a piece of paper glued to the outside of the cylinder $x^2 + y^2 = 1$. Suppose that we open a compass to a radius r ($0 < r < 2$), put the stationary point of the compass at the point $(1, 0, 0)$ on the cylinder, and draw a "circle" on the paper (that is, we use the pencil end of the compass to draw a curve).

If we now remove the paper from the cylinder and draw a coordinate system with the origin at the stationary compass point, the Y axis in the same direction as the original z axis, and the X axis oriented appropriately, we can now consider the "circle" as a plane figure. Find an equation for this figure in the XY-plane.

Solution.

In the three-dimensional coordinate system, the curve will be the intersection of the surfaces $(x-1)^2 + y^2 + z^2 = r^2$ and $x^2 + y^2 = 1$. Simplifying, we get $2 - 2x + z^2 = r^2$.

We know that we can substitute Y for z. To find X, we notice that (because the radius of the cylinder is 1) X is the angle θ from the positive x-axis to the plane through the z-axis containing the point (x, y, z). We can conclude that $x = \cos(\theta) = \cos(X)$ so the desired equation is

$$2 - 2\cos(X) + Y^2 = r^2.$$

Remark. This problem was suggested by David Goldberg, Ann Arbor, MI.)

4. Determine

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin\left(\frac{\pi/2}{k}\right) - \cos\left(\frac{\pi/2}{k}\right) - \sin\left(\frac{\pi/2}{k+2}\right) + \cos\left(\frac{\pi/2}{k+2}\right) \right).$$

Solution.

Let S_n be the nth partial sum:

$$S_n = \sum_{k=1}^{n} \left(\sin\left(\frac{\pi/2}{k}\right) - \cos\left(\frac{\pi/2}{k}\right) - \sin\left(\frac{\pi/2}{k+2}\right) + \cos\left(\frac{\pi/2}{k+2}\right) \right).$$

The first few partial sums are:

$$\begin{aligned}
S_1 &= 1 - \sin(\pi/6) + \cos(\pi/6);\\
S_2 &= S_1 + \sin(\pi/4) - \cos(\pi/4) - \sin(\pi/8) + \cos(\pi/8)\\
&= 1 - \sin(\pi/6) + \cos(\pi/6) - \sin(\pi/8) + \cos(\pi/8);\\
S_3 &= S_2 + \sin(\pi/6) - \cos(\pi/6) - \sin(\pi/10) + \cos(\pi/10)\\
&= 1 - \sin(\pi/8) + \cos(\pi/8) - \sin(\pi/10) + \cos(\pi/10);\\
S_4 &= S_3 + \sin(\pi/8) - \cos(\pi/8) - \sin(\pi/12) + \cos(\pi/12)\\
&= 1 - \sin(\pi/10) + \cos(\pi/10) - \sin(\pi/12) + \cos(\pi/12).
\end{aligned}$$

By induction we have

$$S_n = 1 - \sin\left(\frac{\pi/2}{n+1}\right) - \sin\left(\frac{\pi/2}{n+2}\right) + \cos\left(\frac{\pi/2}{n+1}\right) + \cos\left(\frac{\pi/2}{n+2}\right),$$

and

$$\lim_{n\to\infty} S_n = 1 - 0 - 0 + 1 + 1 = 3.$$

5. Mersenne primes continue to make news. A number $M_p = 2^p - 1$ is a Mersenne prime if and only if p is prime and $2^p - 1$ is also prime. Let the operator DS denote "form the sum of the digits"; for example, $DS(5119) = 16$. Let the operator DR denote "execute DS repeatedly until a result in the interval $[1, 9]$ is obtained"; for example, $DR(5119) = DS^2(5119) = DS(16) = 7$.

(a) Prove the following lemma.

Lemma. If A and B are natural numbers, then

$$DR(AB) = DR(DR(A) \cdot DR(B)).$$

(b) Prove that for any Mersenne prime greater than 7, $DR(M_p) = 1$ or 4. (You may use the Lemma in part (a) without proving part (a)).

Solution.

(a) Let

$$A = \sum_{k=0}^{s} a_k 10^k$$

be a natural number. Then

$$DS(A) = \sum_{k=0}^{s} a_k,$$

so upon subtraction,

$$A - DS(A) = \sum_{k=0}^{s} a_k(10^k - 1) \equiv 0 \pmod 9.$$

If we set $DS(A) = A'$, then $A' - DS(A') \equiv 0 \pmod 9$. Combining this with the previous congruence gives $A - DS^2(A) \equiv 0 \pmod 9$. This process is continued until for some n, one has $DS^n(A) = DR(A)$ and

$$A \equiv DR(A) \pmod 9. \tag{$*$}$$

Similarly, for any other natural number B, $B \equiv DR(B) \pmod 9$, and multiplication of the congruences gives

$$AB \equiv DR(A) \cdot DR(B) \pmod 9. \tag{$**$}$$

Application of $(*)$ to $(**)$ gives, finally,

$$DR(AB) = DR(DR(A) \cdot DR(B)). \tag{$***$}$$

This is an equality, rather than a congruence, since both sides lie in the interval $[1, 9]$.

(b) If N, a natural number, can be written in the form $N = 9a + b$, $1 \le b \le 8$, then eq. $(*)$ shows that $DR(N)$ is of the form $9a' + b$, where $a' = 0$. But whenever $N = 9a + b$, $b = 0$, then from eq. $(*)$ and the definition of $DR(N)$, we always have $DR(N) = 9$. Hence, the equality

$$DR(N - 1) = DR(N) - 1$$

holds if and only if 9 does not divide $n - 1$ (i.e., $9 \nmid (N - 1)$).

Now consider a Mersenne prime, M_p; as $9 \nmid M_p$, then

$$DR(M_p) = DR(M_p + 1) - 1. \qquad (****)$$

All primes p larger than 3 are of the form $6k + 1$ or $6k + 5$.

The first case, together with eq. $(****)$, yields

$$\begin{aligned} DR(M_p) &= DR(2^{6k+1}) - 1 \\ &= DR[DR(64^k) \cdot DR(2)] - 1 \qquad \text{from } (***) \\ &= DR[1^k \cdot 2] - 1 \qquad \text{from } (***)\text{, repeatedly} \\ &= 1. \end{aligned}$$

For the second case ($p = 6k + 5$), we obtain similarly,

$$\begin{aligned} DR(M_p) &= DR(2^{6k+5}) - 1 \\ &= DR[DR(64^k) \cdot DR(32)] - 1 \\ &= DR[1^k \cdot 5] - 1 \\ &= 4. \end{aligned}$$