2012 Missouri Collegiate Mathematics Competition
Session I

1. Two circles in the first quadrant are tangent to each other and both are tangent to the x-axis and the line $y = mx$, where m is a positive constant. Find the ratio of the radius of the larger circle to the radius of the smaller circle as a function of θ, the angle between the x-axis and the line $y = mx$.

2. Prove that $AB - BA \neq I_n$ for any $n \times n$ matrices A and B over the real numbers, where I_n denotes the $n \times n$ identity matrix.

3. Find

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}} \right).$$

4.

Part 1. Let n be a positive integer and let $P_{n-1}(x)$ be the polynomial equal to $P_{n-1}(x) = \frac{x^n - 1}{x - 1}$. Find $P_{n-1}(1)$.

Part 2. On the circle of radius 1, let $V_1, V_2, \ldots, V_n$ be the vertices of a regular n-gon inscribed in the circle. Let $\lambda_k = dist(V_1, V_k)$, $k = 1, 2, \ldots, n$. Show that

$$\prod_{k=2}^{n} \lambda_k = n.$$

Part 3. For the regular n-gon in Part 2, find the product of the lengths of all the line segments joining the vertices.

5. Let S be a set of twelve distinct positive integers such that for distinct a, b, c, and d in S, $a + b \neq c + d$. Prove that the largest element in S is greater than 56.

2012 Missouri Collegiate Mathematics Competition
Session II

1. Express 2012 as a sum of (two or more) consecutive integers.

2. Let f be a function that has a continuous derivative over the interval $[a, b]$, and let $f(a) = f(b) = 0$. Prove that

$$\max_{a \le x \le b} |f'(x)| \ge \frac{4}{(b-a)^2} \int_a^b |f(x)| dx.$$

3. For real $a > 0$ define the sequence $\{x_n\}$ by

$$x_{n+1} = a(x_n^2 + 4), \qquad x_0 = 0.$$

Determine necessary and sufficient conditions on a for $\lim_{n \to \infty} x_n$ to exist and be finite.

4. Let $P(x)$ be a polynomial in x of degree $n > 2$. Suppose that $a \ne b$, $P(a) = n_1$, and $P(b) = n_2$. What is the remainder $R(x)$ when $P(x)$ is divided by $(x-a)(x-b)$?

5. Let S be the set of points in $\mathbb{R}^2$ that constitute the graph of $y = x^2$, $-2 \le x \le 2$, and let $d(p_1, p_2)$ denote the Euclidean distance between $p_1, p_2 \in S$. Determine p_1 and p_2 that maximizes $d(p_1, p_2)$.