2013 Missouri Collegiate Mathematics Competition
Session I

1. Consider the sets of consecutive integers $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, $\{7,8,9,10\}$, $\ldots$, where each set contains one more element than the preceding one, and where the first element of each set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Find S_{32}.

2. Find all positive real solutions x to the equation

$$4\lfloor x \rfloor = 3x\{x\},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$.

3. Evaluate the following where $p > 1$.

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}.$$

4. Neither *Mathematica* nor *Maple* can find the exact value of the following definite integral. Can you? We think you can. Do it!

$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\,dx.$$

5. Define the sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2,$$

and for $n \geq 3$,

$$a_{n+1} = \frac{n a_n a_{n-2}}{a_{n-1}}.$$

(a) Prove that a_n is a positive integer for all $n \geq 1$.

(b) Define the sequence $\{b_n\}_{n=1}^{\infty}$ by

$$b_n = \frac{a_n}{\sqrt{(n+1)!}} \quad \text{for} \quad n \geq 1.$$

Prove that $\{b_n\}$ is bounded.

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Session II

1. Let r be a real number with $0 < r \leq 1$. Let $S(r)$ be the region in the (x, y)-plane bounded by the curve $y = x^r$, the x-axis, and the line $x = 1$. Let $G(r)$ denote the centroid of $S(r)$. Let ℓ_r be the line that passes through the origin and $G(r)$. Let $T(r)$ be the region in the (x, y)-plane bounded by ℓ_r, the x-axis, and the line $x = 1$. Find the value of r that minimizes the ratio of the area of $T(r)$ to the area of $S(r)$.

2. Let

$$f(r) = \sum_{j=2}^{2013} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2013^r}.$$

Find

$$\sum_{k=2}^{\infty} f(k).$$

3. Define an *Egyptian triangle* to be one that is similar to a 3-4-5 right triangle. A square sheet of paper is folded so that one corner touches the midpoint of the opposite side. The folded sheet will thus be composed of three triangles "one sheet thick" and a quadrilateral "two sheets thick." Prove that all three triangles are Egyptian triangles.

4. Prove that $\frac{x^2+y^2}{4} \leq e^{x+y-2}$ for $x \geq 0$ and $y \geq 0$.

5. A point (x, y) is chosen at random from the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ (uniform distribution on the unit square). Find the probability that x and y are side lengths of an isosceles triangle of perimeter at most 1.